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RATIONALLY CONNECTED 3-FOLDS AND SYMPLECTIC GEOMETRY

by

Claire Voisin

Pour Jean Pierre Bourguignon, à l'occasion de ses 60 ans.

Abstract. — We study the following question asked by Kollár: Let X be a rationally connected 3-fold, and Y be a compact Kähler 3-fold symplectically equivalent to it. Is Y rationally connected? We show that the answer is positive if X is Fano or $b_2(X) \leq 2$.

***Résumé* (3-variétés rationnellement connexes et géométrie symplectique).** — Nous étudions la question suivante posée par Kollár: soient X et Y des variétés kählériennes compactes de dimension 3 symplectiquement équivalentes. On suppose que X est rationnellement connexe. Y est-elle aussi rationnellement connexe? Nous montrons que la réponse est positive si X est une variété de Fano ou $b_2(X) \leq 2$.

0. Introduction

Let X be a compact Kähler manifold. Denoting by J the operator of complex structure acting on T_X, Kähler forms on X are symplectic forms which satisfy the compatibility conditions

$$\omega(Ju, Jv) = \omega(u, v),\ u,\ v \in T_{X,x},\ \omega(u, Ju) > 0,\ 0 \neq u \in T_{X,x}.$$

The first condition tells that ω is of type $(1,1)$. The last condition is called the taming condition. The set of Kähler forms is a convex cone, in particular connected, and thus determines a deformation class of symplectic forms on X.

Let X and Y be two complex projective or compact Kähler manifolds. We will say that X and Y are symplectically equivalent if for some symplectic forms α on X, resp. β on Y, which are in the deformation class of a Kähler form on X, resp. Y, there is a diffeomorphism

$$\psi : X \cong Y,$$

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such that $\psi^*\beta = \alpha$. Notice that ψ^* induces a bijection between the sets of symplectic forms which are in the deformation class of a symplectic form on Y and X, and thus we may assume that α is a taming form, or even a Kähler form on X.

In the sequel, the compact Kähler manifolds X we will consider are *uniruled* manifolds, which means the following (cf [**8**]):

Definition 0.1. — *A projective complex manifold (or compact Kähler) is uniruled if there exist compact complex manifolds Z and B, and dominating morphisms*

$$f : Z \to X,\ g : Z \to B,$$

where f is non constant on the fibers of g and the generic fiber of g is isomorphic to $\mathbb{P}^1$.

In other words, there is a (maybe singular) rational curve in X passing through any point of X, where a (singular) rational curve is defined as a connected curve whose normalization has only rational components.

The starting point of this work is the following result, due independently to Kollár [**9**] and Ruan [**19**] (we refer to [**6**], [**13**], [**14**] for purely symplectic characterizations and studies of uniruledness) :

Theorem 0.2. — *Let X and Y be two symplectically equivalent compact Kähler manifolds. Then if X is uniruled, Y is also uniruled.*

We sketch later on the proof of this result, in order to point out why the proof does not extend to cover the rational connectedness property, which we will consider in this paper. Let us recall the definition (cf [**2**], [**10**], [**8**]).

Definition 0.3. — *A compact Kähler manifold X is rationally connected if for any two points x, $y \in X$, there exists a (maybe singular) rational curve $C \subset X$ with the property that $x \in C$, $y \in C$.*

Examples of rationally connected varieties are given by smooth Fano varieties, i.e. smooth projective varieties X satisfying the condition that $-K_X$ is ample. (This is the main result of [**2**], and [**10**].)

The following conjecture appears in [**9**]. It was asked to me by Pandharipande and Starr :

***Conjecture 0.4* (Kollár)**. — *Assume X is rationally connected. Let Y be a compact Kähler manifold symplectically equivalent to X. Then Y is also rationally connected.*

Remark 0.5. — A compact Kähler manifold X whis is rationally connected satisfies $H^2(X, \mathcal{O}_X) = 0$, hence is projective. Thus, under the assumption above, X is projective, and if the answer to conjecture 0.4 is positive, Y is also projective.

This conjecture has an easy positive answer in the case of surfaces, as an immediate consequence of theorem 0.2. Indeed, let X be rationally connected of dimension 2, and let Y be symplectically equivalent to X. Then Y is uniruled, as X is. On the other hand $b_1(Y) = 0$, because $b_1(X) = 0$ and Y is diffeomorphic to X. Thus Y is a rational surface, hence rationally connected.

In this note, we prove the following partial results concerning conjecture 0.4 in dimension 3. I should mention here that in these form the results are partly due to Jason Starr. Indeed, in the original version of this paper, I had worked with a more restricted notion of symplectic equivalence between compact Kähler manifolds, where I considered only symplectic diffeomeorphisms $(X, \alpha) \cong (Y, \beta)$ where α *and* β were taming for the complex structure. Jason Starr showed me how to make the proof of proposition 0.6 work as well when only α is taming, and β is any symplectic form which is a deformation (through a family of symplectic forms) of a Kähler form on Y.

Proposition 0.6. — *Let X be rationally connected of dimension 3, and let Y be compact Kähler symplectically equivalent to X. If Y is not rationally connected, X and Y admit almost holomorphic rational maps*

$$\phi : X \dashrightarrow \Sigma,\ \phi' : Y \dashrightarrow \Sigma'$$

to a surface, with rational fibers C, resp. D, of the same homology class (where we use the symplectomorphism $\psi : X \cong Y$ giving symplectic equivalence to identify $H_2(X, \mathbb{Z})$ and $H_2(Y, \mathbb{Z})$).

Here *almost holomorphic* means that the map is well-defined near a generic fiber. We then consider the case where the above map ϕ is well-defined.

Proposition 0.7. — *Under the same assumptions as in proposition 0.6, assume that the rational map ϕ above is well-defined and that either Σ is smooth, or ϕ does not contract a divisor to a point. Then Y is also rationally connected.*

We will use this result together with some birational geometry arguments to prove the following:

Theorem 0.8. — *Let X, Y be compact Kähler 3-folds. Assume that X and Y are symplectically equivalent and that one of the two following assumptions hold:*

1. *X is Fano.*
2. *X is rationally connected, and $b_2(X) \leq 2$.*

Then Y is rationally connected.

This anwers conjecture 0.4 when X is a Fano threefold or satisfies $b_2 \leq 2$. The two considered cases have a small overlap. In the class where $b_2(X) \leq 2$, one has all the blow-ups of Fano manifolds with $b_2 = 1$ along a connected submanifold. Thus this is not a bounded family. It is known on the contrary that Fano manifolds form a bounded family (see [**2**], [**10**], or [**17**] for the 3-dimensional case). However the bound for b_2 of a Fano threefold is 10 (cf [**17**]), showing that the Fano case is far from being included in the second case.

Remark 0.9. — Note that for varieties with $b_2 = 1$, conjecture 0.4 obviously has an affirmative answer. Indeed a uniruled projective manifold with $b_2 = 1$ is necessarily Fano. Hence if X is rationally connected with $b_2 = 1$, by theorem 0.2 any projective manifold which is symplectomorphic to it is also uniruled with $b_2 = 1$, hence Fano, hence rationally connected.

Remark 0.10. — The results presented here have a partial overlap with [**3**], where the authors show that for rigid and "primitive" Fano threefolds with $b_2 = 2$ and $b_3 = 0$, the projective (equivalently Kähler) complex structure is unique. I thank the referee for bringing this reference to my attention.

To conclude this introduction, let us sketch the proof of theorem 0.2, and explain on an example the difficulty one meets to extend it to the rational connectedness question.

Proof of theorem 0.2. — Let α be a taming symplectic form on X (one can take here a Kähler form). We will denote in the sequel the degree of curves C in X with respect to α (that is the integrals $\int_C \alpha$) by $\deg_\alpha(C)$. Let $\mu_\alpha(X)$ be the minimum of the following set:

$$S_X := \{\deg_\alpha(C),\ C \text{ moving rational curve in } X\}.$$

Here by "moving", we mean that the deformations of C sweep-out X. Note that the minimum of the set S_X is well defined, because there are finitely many families of curves of bounded degree in X and the $(1,1)$-part $\alpha^{1,1}$ of α is $> \epsilon\omega$ where ω is any Kähler form on X. Let now C be a moving rational curve on X, which satisfies $\deg_\alpha(C) = \mu_\alpha(X)$ and let $[C] \in H_2(X, \mathbb{Z})$ be its homology class. We claim that for $x \in X$, and for adequate cohomology classes $A_1, \ldots, A_r \in H^4(X, \mathbb{Z})$, the Gromov-Witten invariant $GW_{0,[C]}([x], A_1, \ldots, A_r)$ counting genus 0 curves passing through x and meeting representatives B_i of the homology classes Poincaré dual to A_i, is non zero. To see this, we observe that by minimality of $\deg_\alpha(C)$, any genus 0 curve of degree $< \deg_\alpha(C)$ is not moving, that is, its deformations do not sweep-out X. It follows that for a general point $x \in X$, any genus 0 curve of class $[C]$ and passing through x is irreducible, with normal bundle generated by sections. This implies that the set $Z_{x,[C]}$ of rational curves of classes $[C]$ passing through x has the expected dimension and it is nonempty by assumption. Let r be its dimension, and choose for A_i, $1 \leq i \leq r$, a class h^2, where h is a Kähler class on X. It is then clear