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ON THE NUMBER OF MINIMAL SURFACES WITH A GIVEN BOUNDARY

by

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Dedicated to Jean Pierre Bourguignon on the occasion of his 60th birthday

Abstract. — We prove results allowing us to count, mod 2, the number of embedded minimal surfaces of a specified topological type bounded by a curve $\Gamma \subset \partial N$, where N is a weakly mean convex 3-manifold with piecewise smooth boundary. These results are extended to curves and minimal surfaces with prescribed symmetries. The parity theorems are used in an essential manner to prove the existence of embedded genus-g helicoids in $\mathbf{S}^2 \times \mathbf{R}$, and we give an outline of this application.

***Résumé* (Sur le nombre de surfaces minimales avec une frontière donnée).** — Nous démontrons des résultats qui nous permettent de compter, modulo 2, le nombre de surfaces minimales plongées d'un type topologique donné, borné par une courbe $\Gamma \subset \partial N$, où N est une 3-variété convexe faiblement moyenne munie d'une frontière lisse par morceaux. Ces résultats sont étendus aux courbes et aux surfaces minimales à symétries préscrites. Les théorèmes de parité sont utilisés de manière essentielle pour prouver l'existence d'hélicoïdes de genre imbriqué g dans $\mathbf{S}^2 \times \mathbf{R}$, et nous donnons un aperçu de cette application.

1. Introduction

In [**4**], Tomi and Tromba used degree theory to solve a longstanding problem about the existence of minimal surfaces with a prescribed boundary: they proved that every smooth, embedded curve on the boundary of a convex subset of $\mathbf{R}^3$ must bound an embedded minimal disk. Indeed, they proved that a generic such curve must bound an odd number of minimal embedded disks. White [**8**] generalized their result by proving the following parity theorem. Suppose N is a compact, strictly convex domain in $\mathbf{R}^3$

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with smooth boundary. Let Σ be a compact 2-manifold with boundary. Then a generic smooth curve $\Gamma \cong \partial\Sigma$ in ∂N bounds an odd or even number of embedded minimal surfaces diffeomorphic to Σ according to whether Σ is or is not a union of disks.

In this paper, we generalize the parity theorem in several ways. First, we prove (Theorem 2.1) that the parity theorem holds for any compact riemannian 3-manifold N such that N is strictly mean convex, N is homeomorphic to a ball, ∂N is smooth, and N contains no closed minimal surfaces. We then further relax the hypotheses by allowing N to be mean convex rather than strictly mean convex, and to have piecewise smooth boundary. Note that if N is mean convex but not strictly mean convex, then Γ might bound minimal surfaces that lie in ∂N. We prove (Theorem 2.4) that the parity theorem remains true for such N provided (1) unstable surfaces lying in ∂N are not counted, and (2) no two contiguous regions of $(\partial N) \setminus \Gamma$ are both smooth minimal surfaces. We give examples showing that the theorem is false without these provisos.

We extend the parity theorem yet further (see Theorem 2.7) by showing that, under an additional hypothesis, it remains true for minimal surfaces with prescribed symmetries.

The parity theorems described above are all mod 2 versions of stronger results that describe integer invariants. The stronger results are given in section 3.

The parity theorems are used in an essential way to prove the the existence of embedded genus-g helicoids in $\mathbf{S}^2 \times \mathbf{R}$. In Sections 4 and 5 we give a very brief outline of this application. (The full argument will appear in [**3**].)

2. Counting minimal surfaces

Throughout the paper, N will be a compact riemannian 3-manifold and Σ will be a fixed compact 2 manifold. If Γ is an embedded curve in N diffeomorphic to $\partial\Sigma$, we let $\mathcal{M}(N,\Gamma)$ denote the set of embedded minimal surfaces in N that are diffeomorphic to Σ and that have boundary Γ. We let $|\mathcal{M}(N,\Gamma)|$ denote the number of surfaces in $\mathcal{M}(N,\Gamma)$.

In case N has smooth boundary, we say that N is strictly mean convex provided the mean curvature is a (strictly) positive multiple of the inward unit normal on a dense subset of ∂N.

2.1. Theorem. — *Let N be a smooth, compact, strictly mean convex riemannian 3-manifold that is homeomorphic to a ball and that has smooth boundary. Suppose also that N contains no closed minimal surfaces. Let $\Gamma \subset \partial N$ be a smooth curve diffeomorphic to $\partial\Sigma$. Assume that Γ is bumpy in the sense that no surface in $\mathcal{M}(N,\Gamma)$ supports a nontrivial normal Jacobi field with zero boundary values.*

Then $|\mathcal{M}(N,\Gamma)|$ *is even unless* Σ *is a union of disks, in which case* $|\mathcal{M}(N,\Gamma)|$ *is odd.*

We remark that generic smooth curves $\Gamma \subset \partial N$ are bumpy [**7**].

Proof. — Theorems 2.1 and 2.3 of [**8**] are special cases of the theorem. The proofs given there establish the more general result here provided one makes the following observations:

1. There N was assumed to be strictly convex, but exactly the same proof works assuming strict mean convexity.
2. There Σ was assumed to be connected, but the same proof works for disconnected Σ.
3. In the proofs of Theorems 2.1 and 2.3 of [**8**], the assumption that N is a subset of $\mathbf{R}^3$ was used in order to invoke an isoperimetric inequality, i.e., an inequality bounding the area of a minimal surface in N in terms of the length of its boundary. There are compact mean convex 3-manifolds for which no such isoperimetric inequality holds. However, if (as we are assuming here) N contains no closed minimal surfaces, then N does admit such an isoperimetric inequality [**9**].
4. In the proofs in [**8**], one needs to isotope any specified component of Γ to a curve C that bounds exactly one minimal surface, namely an embedded disk. This was achieved by choosing C to be a planar curve. For a general ambient manifold N, "planar" makes no sense. However, any sufficiently small, nearly circular curve $C \subset \partial N$ bounds exactly one embedded minimal disk and no other minimal surfaces. (This property of such a curve C is proved in the last paragraph of §3 in [**8**].) □

2.2. Mean convex ambient manifolds N with piecewise smooth boundary. — For the remainder of the paper, we allow ∂N to be piecewise smooth. For simplicity, let us take this to mean that ∂N is a union of smooth 2-manifolds with boundary ("faces" of N), any two of which are either disjoint or meet along a common edge with interior angle everywhere strictly between 0 and 2π. (More generally, one could allow the faces of N to have corners.) We say that such an N is mean convex provided (1) at each interior point of each face of N, the mean curvature vector is a nonnegative multiple of the inward-pointing unit normal, and (2) where two faces meet along an edge, the interior angle is everywhere at most π.

The following example shows what can go wrong in Theorem 2.1 if N is mean convex but not strictly mean convex.

Example 1. Let N be a region in $\mathbf{R}^3$ whose boundary consists of an unstable catenoid C bounded by two circles, together with the two disks bounded by those circles. Note that N is mean convex with piecewise smooth boundary. Let Γ be a pair of horizontal circles in C that are bumpy (in the sense of Theorem 2.1). Theorem 2.1 suggests that Γ should bound an even number of embedded minimal annuli in N. First consider the case when Γ consists of two circles in C very close to the waist circle. Then Γ bounds precisely two minimal annuli. One of them is the component of C bounded by Γ. Because the circles in Γ are close, this annulus is strictly stable. The other annulus bounded by Γ is a strictly unstable catenoid lying in the interior of N. In order to get an even number of examples, we must count the stable catenoid lying on C. Now suppose the two components of Γ are the two components of ∂C. Then again Γ bounds exactly two minimal annuli: the unstable catenoid C, which is part of ∂N, and a strictly stable catenoid that lies outside N. Here, of course, we do not count the stable catenoid since it does not lie in N. Thus to get an even number, we also must not count the unstable catenoid that lies in ∂N.

This example motivates the following definition:

2.3. Definition. — $\mathcal{M}^*(N,\Gamma)$ is the set of embedded minimal surfaces $M \subset N$ such that

i.) $\partial M = \Gamma$,
ii.) M is diffeomorphic to Σ, and
iii.) each connected component of M lying in ∂N is stable.

Example 1 suggests that in order to generalize Theorem 2.1 to mean convex N with piecewise smooth boundary, we should replace $\mathcal{M}(N,\Gamma)$ by $\mathcal{M}^*(N,\Gamma)$. However, even if one makes that replacement, the following example shows that an additional hypothesis is required.

Example 2. Let N be a compact, convex region in $\mathbf{R}^3$ such that ∂N is smooth and contains a planar disk D. Let Γ be a pair of concentric circles lying in D. Then Γ bounds exactly one minimal annulus: the region in D between the two components of Γ. That annulus is strictly stable and lies in ∂N. Thus Γ is bumpy (in the sense of Theorem 2.1) and $|\mathcal{M}^*(N,\Gamma)| = 1$. Consequently, if we wish $|\mathcal{M}^*(N,\Gamma)|$ to be even (as Theorem 2.1 suggests it should be), then we need an additional hypothesis on N and Γ.

Note that in example 2, $(\partial N)\setminus\Gamma$ contains two contiguous connected components (a planar annulus and a planar disk) both of which are minimal surfaces. The additional hypothesis we require is that $(\partial N)\setminus\Gamma$ contains no two such components.

2.4. Theorem. — *Let N be a smooth, compact, mean convex riemannian 3-manifold that is homeomorphic to a ball, that has piecewise smooth boundary, and that contains*