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THE PROJECTIVE HULL OF CERTAIN CURVES IN $\mathbb{C}^2$

by

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Dedicated to Jean Pierre Bourguignon
on the occasion of his sixtieth birthday

Abstract. — The projective hull $\widehat{X}$ of a compact set $X \subset \mathbb{P}^n$ is an analogue of the classical polynomial hull of a set in $\mathbb{C}^n$. In the special case that $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, the affine part $\widehat{X} \cap \mathbb{C}^n$ can be defined as the set of points $x \in \mathbb{C}^n$ for which there exists a constant M_x so that

$$\left|p(x)\right| \leq M_x^d \sup_X |p|$$

for all polynomials p of degree $\leq d$, and any $d \geq 1$. Let $\widehat{X}(M)$ be the set of points x where M_x can be chosen $\leq M$. Using an argument of E. Bishop, we show that if $\gamma \subset \mathbb{C}^2$ is a compact real analytic curve (not necessarily connected), then for any linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, the set $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *is finite for almost all* $z \in \mathbb{C}$. It is then shown that for any compact stable real-analytic curve $\gamma \subset \mathbb{P}^n$, the set $\widehat{\gamma} - \gamma$ *is a* 1*-dimensional complex analytic subvariety of* $\mathbb{P}^n - \gamma$. Boundary regularity for $\widehat{\gamma}$ is also discussed in detail.

***Résumé* (L'enveloppe projective de certaines courbes dans $\mathbb{C}^2$).** — L'enveloppe projective $\widehat{X}$ d'un compact $X \subset \mathbb{P}^n$ est l'analogue de l'enveloppe polynomiale classique d'un sous-ensemble de $\mathbb{C}^n$. Dans le cas particulier où $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, la partie affine $\widehat{X} \cap \mathbb{C}^n$ peut être définie en tant qu'ensemble de points $x \in \mathbb{C}^n$ pour lesquels il existe une constante M_x telle que

$$\left|p(x)\right| \leq M_x^d \sup_X |p|$$

pour tous les polynômes p de degré $\leq d$, et tout $d \geq 1$. Soit $\widehat{X}(M)$ l'ensemble de points x où M_x peut être choisi $\leq M$. En utilisant un argument d'E. Bishop, nous montrons que si $\gamma \subset \mathbb{C}^2$ est une courbe analytique réelle compacte (non nécessairement connexe), alors pour toute projection linéaire $\pi : \mathbb{C}^2 \to \mathbb{C}$, l'ensemble $\widehat{\gamma}(M) \cap \pi^{-1}(z)$ *est fini pour presque tout* $z \in \mathbb{C}$. Nous montrons alors que pour toute courbe analytique réelle compacte stable $\gamma \subset \mathbb{P}^n$, l'ensemble $\widehat{\gamma} - \gamma$ *est une sous-variété de* $\mathbb{P}^n - \gamma$ *analytique complexe de dimension* 1. Nous discutons également en détail la régularité de la frontière de $\widehat{\gamma}$.

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1. Introduction

The classical *polynomial hull* of a compact subset X of $\mathbb{C}^n$ is the set of points $x \in \mathbb{C}^n$ such that

$$|p(x)| \le \sup_X |p| \quad \text{for all polynomials } p. \tag{1.1}$$

In [**4**] the first two authors introduced an analogue for compact subsets of projective space. Given $X \subset \mathbb{P}^n$, the *projective hull of* X is the set $\widehat{X}$ of points $x \in \mathbb{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|P(x)\| \le C_x^d \sup_X \|P\| \quad \text{for all sections } P \in H^0\big(\mathbb{P}^n, \mathcal{O}(d)\big) \tag{1.2}$$

and all $d \ge 1$. Here $\mathcal{O}(d)$ is the d-th power of the hyperplane bundle with its standard metric. Recall that $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is given naturally as the set of homogeneous polynomials of degree d in homogeneous coordinates. If X is contained in an affine chart $X \subset \mathbb{C}^n \subset \mathbb{P}^n$ and $x \in \mathbb{C}^n$, then condition (1.2) is equivalent to

$$|p(x)| \le M_x^d \sup_X |p| \quad \text{for all polynomials } p \text{ of degree } d \tag{1.3}$$

and all $d \ge 1$ where $M_x = \rho\sqrt{1 + \|x\|^2} C_x$ and ρ depends only on X. Therefore the set $\widehat{X} \cap \mathbb{C}^n$ consists exactly of those points $x \in \mathbb{C}^n$ for which there exists an M_x satisfying condition (1.3).

This paper is concerned with the case where $X = \gamma$ is a real analytic curve. In [**4**] evidence was given for the following conjecture.

Conjecture 1.1. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of simple closed real analytic curves. Then $\widehat{\gamma} - \gamma$ is a 1-dimensional complex analytic suvariety of $\mathbb{P}^n - \gamma$.*

This conjecture has many interesting geometric consequences (see [**7**], [**5**], and [**6**]). The assumption of real analyticity is important. The conjecture does not hold for all smooth curves. In particular, it does not hold for curves which are not pluripolar.

One point of this paper is to prove Conjecture 1.1 under the hypothesis that the function C_x is bounded on $\widehat{\gamma}$. We begin by adapting arguments of E. Bishop [**2**] to prove the following finiteness theorem.

Theorem 1.1. — *Let $\gamma \subset \mathbb{C}^2$ be a finite union of simple closed real analytic curves. Set*

$$\widehat{\gamma}_M \equiv \{x \in \widehat{\gamma} \cap \mathbb{C}^2 : M_x \le M\}$$

where M_x is the function appearing in condition (1.3). Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection. Then

$$\widehat{\gamma}_M \cap \pi^{-1}(z) \quad \textit{is finite for almost all } z \in \mathbb{C}.$$

Consequently, $\widehat{\gamma} \cap \pi^{-1}(z)$ is countable for almost all $z \in \mathbb{C}$.

In Section 3 this theorem is combined with results from [**4**] and the theorems concerning maximum modulus algebras to prove the following.

A set $X \subset \mathbb{P}^n$ is called *stable* if the function C_x in (1.2) is bounded on $\widehat{X}$.

Note that if X is stable and $X \subset \mathbb{C}^n \subset \mathbb{P}^n$, then the function M_x is bounded on $\mathbb{C}^n$ by $\rho\sqrt{1+\|x\|^2}$.

Theorem 1.2. — *Let $\gamma \subset \mathbb{P}^n$ be a finite union of simple closed real analytic curves. Assume γ is stable. Then $\widehat{\gamma}-\gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{P}^n - \gamma$.*

2. The finiteness theorem

Let X be a compact set in $\mathbb{C}^n$ and denote by $\mathcal{P}_d$ the space of polynomials of degree $\leq d$ on $\mathbb{C}^n$.

Definition 2.1. — Denote by $\widehat{X} \cap \mathbb{C}^n$ the set of all $x \in \mathbb{C}^n$ such that there exists a constant M_x with

$$|P(x)| \leq M_x^d \sup_X |P| \tag{2.1}$$

for every $P \in \mathcal{P}_d$ and $d \geq 1$. The set $\widehat{X} \cap \mathbb{C}^n$ is called *the projective hull of X in* $\mathbb{C}^n$.

As noted above, the projective hull, defined in [**4**], is a subset of projective space $\mathbb{P}^n$, and the set $\widehat{X} \cap \mathbb{C}^n$ is exactly that part of the projective hull which lies in the affine chart $\mathbb{C}^n \subset \mathbb{P}^n$. Closely related to Definition 2.1 is the following.

Definition 2.2. — Fix a number $M \geq 1$ and a point $z \in \mathbb{C}^{n-1}$. Then we set

$$\widehat{X}_M(z) = \{w \in \mathbb{C} : |P(z,w)| \leq M^d \sup_X |P|, \ \forall P \in \mathcal{P}_d \text{ and } \forall d \geq 1\}$$

and let $\widehat{X}(z) = \bigcup_{M\geq 1} \widehat{X}_M(z) = \{w \in \mathbb{C} : (z,w) \in \widehat{X}\}$.

We consider a special case of these definitions. We fix $n = 2$ and consider a simple closed real-analytic curve X in $\mathbb{C}^2$. Let Δ denote the unit disk in $\mathbb{C}$.

Theorem 2.1. — *Fix $M \geq 1$. For almost all $z \in \Delta$, $\widehat{X}_M(z)$ is a finite set.*

Corollary 2.1. — *For almost all $z \in \mathbb{C}$ the set $\widehat{X}(z)$ is countable.*

We shall prove Theorem 2.1 by adapting an argument, for the case of polynomially convex hulls, by Errett Bishop in [**2**]. We shall follow the exposition of Bishop's argument in [**10**, Chap. 12].

Definition 2.3. — The polynomial $Q(z,w) = \sum_{n,m} c_{nm} z^n w^m$ is called a *unit polynomial* if $\max_{n,m} |c_{nm}| = 1$.

Definition 2.4. — The polynomial $Q(z,w) = \sum_{n,m} c_{nm} z^n w^m$ is said to have *bidegree* (d,e), for non-negative integers d and e, if $c_{nm} = 0$ unless $n \le d$ and $m \le e$, and d, e are minimal with this property.

Note that $\deg Q \le d + e \le 2 \deg Q$.

Definition 2.5. — Fix $M \ge 1$. For each $z \in \mathbb{C}$ set

$$S_M(z) = \{w \in \mathbb{C} : |Q(z,w)| \le (M^{d+e}) \sup_X |Q|, \quad \forall Q \in \mathbb{C}[z,w] \text{ of bidegree } (d,e) \text{ for } d,e \ge 1\}.$$

We now fix a number $M \ge 1$ and keep it fixed throughout what follows.

Theorem 2.2. — *For almost all $z \in \Delta$, $S_M(z)$ is a finite set.*

Theorem 2.1 is an immediate consequence of Theorem 2.2. To see this, fix $z \in \Delta$ and choose $w \in \widehat{X}_M(z)$. Choose next a polynomial Q of bidegree (d,e) and let $\delta = \deg Q$. Then

$$\big|Q(z,w)\big| \le M^\delta \|Q\|_X \le M^{d+e} \|Q\|_X$$

and so $w \in S_M(z)$. Since this holds for all such w, $\widehat{X}_M(z) \subseteq S_M(z)$. By Theorem 2.2 $S_M(z)$ is a finite set for a. a. $z \in \Delta$, so $\widehat{X}_M(z)$ is a finite set for almost all $z \in \Delta$. Thus Theorem 2.1 holds.

We now go to the proof of Theorem 2.2.

Lemma 2.1. — *Let Ω be a plane domain, let K be a compact set in Ω, and fix $z_0 \in \Omega$. Then there exists a constant r, $0 < r < 1$, so that if f is holomorphic on Ω and $|f| < 1$ on Ω and if f vanishes to order λ at z_0, then $|f| \le r^\lambda$ on K.*

Proof. — We construct a bounded and smoothly bounded subdomain Ω_0 of Ω with $\overline{\Omega}_0 \subset \Omega$, $z_0 \in \Omega_0$ and $K \subset \Omega_0$. Denote by $G(z_0, z)$ the Green's function of Ω_0 with pole at z_0.

Then $\mathrm{e}^{-(G+iH)}$ is a multiple-valued holomorphic function on Ω_0 with a single-valued modulus e^{-G}, and this modulus is $= 1$ on $\partial\Omega_0$ (H is the harmonic conjugate of G). Consequently,

$$f/\mathrm{e}^{-\lambda(G+iH)}$$

is multiple-valued and holomorphic on Ω_0, and its modulus is single-valued and < 1 on $\partial\Omega_0$. By the maximum principle for holomorphic functions, for each $z \in K$, we have $|f/\mathrm{e}^{-\lambda(G+iH)}| < 1$ at z and so

$$|f(z)| \le \left[\mathrm{e}^{-G(z_0,z)}\right]^\lambda.$$

Putting $r = \sup_K \mathrm{e}^{-G}$, we get our desired inequality. □