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A SURVEY OF THE HYPOELLIPTIC LAPLACIAN

by

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À Jean Pierre Bourguignon pour son soixantième anniversaire

Abstract. — The purpose of this paper is to review the construction of the hypoelliptic Laplacian, in the context of de Rham theory for smooth manifolds, and also the construction of the hypoelliptic Dirac operator in the context of complex Kähler manifolds.

***Résumé* (Compte-rendu sur le laplacien hypoelliptique).** — Le but de cet article est d'établir un compte-rendu de la construction du laplacien hypoelliptique dans le contexte de la théorie de de Rham des variétés lisses, ainsi que de la construction de l'opérateur de Dirac hypoelliptique dans le contexte des variétés kähleriennes complexes.

Introduction

The purpose of this survey is to review certain aspects of the construction of the hypoelliptic Laplacian, in de Rham and in Dolbeault theory. The hypoelliptic Laplacian was introduced in [**3**] in de Rham theory, and in [**5**] for Dirac operators. The crucial analytic foundations for the theory were developed by Lebeau and ourselves in [**8**].

One motivation given in [**3**] is to interpret the hypoelliptic Laplacian in de Rham theory as a semiclassical limit of the Witten deformation of the Hodge theory of the loop space of a Riemannian manifold, which is associated with the energy functional. This point of view remains formal, since the Hodge theory of the loop space of a manifold is not analytically well defined. The motivation for the construction of the hypoelliptic Dirac operator of [**5**] is to understand the effect of replacing the standard

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L_2 metric on the loop space of a manifold by a H^1 metric. Again these considerations remain formal, although ultimately the hypoelliptic Dirac operator is well defined.

Whatever the motivations, and there are many others, some of which are explained in [**4, 6**], the conclusion is that a geometric Laplacian can be deformed into a family of hypoelliptic second order differential operators acting on the total space of the tangent or the cotangent bundle of the given manifold, which interpolates in the proper sense between the Laplacian and the generator of the geodesic flow. The existence of this deformation is counter-intuitive, since ellipticity is a stable property. However, the fact that the hypoelliptic Laplacian acts on a bigger space than the original elliptic Laplacian explains why ultimately it can be made to 'collapse' on the elliptic Laplacian.

Let us finally mention that up to lower order terms, the hypoelliptic Laplacian is the sum of a harmonic oscillator acting in the directions of the fibre, and of the vector field which generates the geodesic flow, these two operators being adequately scaled.

In this paper, first, we fully develop the theory in the case where the base manifold is the circle. The main point is that while in this case, the geometry is trivial, a complete understanding of the hypoelliptic Laplacian and of the interpolation property can be easily obtained via Fourier analysis on the circle and the spectral theory of the harmonic oscillator. The case of the circle is also useful, because the objects which appear there turn out to be at the same time the principal symbols of the geometric hypoelliptic operators, and because the circle is the model of a closed geodesic. The fact that the hypoelliptic Laplacian is self-adjoint with respect to a symmetric form of signature (∞, ∞) appears also naturally in that context.

The basic difference between the case of the circle and the geometric case is that the analysis of the hypoelliptic Laplacian is no longer explicit, and also that the convergence arguments, which are easy for the circle, are built on a functional analytic machinery described in detail in our work with Lebeau [**8**].

Also we describe the construction of the hypoelliptic Laplacian, in the de Rham case, and also for Kähler manifolds. We emphasize the role of the symmetric bilinear forms, at least in the de Rham case, because of the important spectral theoretic consequences which are derived in [**8**].

This paper is organized as follows. In section 1, we consider the case of the circle. Since the hypoelliptic Laplacian is ultimately obtained as a Hodge Laplacian with respect to an exotic bilinear form on the de Rham or the Dolbeault complex, this point of view is systematically emphasized in this simple case too.

In section 2, we recall classical results on the Hodge theory of a compact manifold, and on the Witten deformation of classical Hodge theory which is associated with a smooth function. Also we show that if (M, ω) is a symplectic manifold, there is a symplectic Witten Laplacian, which turns out to be the Lie derivative operator associated with the corresponding Hamiltonian vector field. This point of view is further developed in [**3**], where the hypoelliptic Laplacian in de Rham theory is obtained by linearly interpolating between the Riemannian metric of the base manifold, and the symplectic form of its cotangent bundle.

In section 3, we explain the construction of the hypoelliptic Laplacian in de Rham theory. We also give the main arguments of [**3**] in favour of the fact that the hypoelliptic Laplacian interpolates between the Hodge Laplacian and the geodesic flow.

In section 4, we give the construction of the hypoelliptic Dirac operator of [**5**] in the context of Kähler manifolds, and we give the arguments showing that this operator should indeed be a deformation of the classical elliptic Dirac operator.

As we already said, the analytic justifications which make that the whole construction ultimately exists as a mathematical theory are developed in detail in our work with Lebeau [**8**]. Also applications to Ray-Singer torsion [**19**] and Quillen metrics [**17**] are given in [**8**] and [**5**].

1. The case of the circle

The purpose of this section is to construct the hypoelliptic Laplacian in the case where the base manifold X is just S^1. In this case, all the objects are simple and natural. Besides, the operators which are obtained in this case can be viewed as the symbols of the operators which are obtained later in the geometric case.

This section is organized as follows. In subsection 1.1, we recall elementary properties of elliptic and hypoelliptic operators.

In subsection 1.2, we introduce the Kolmogorov operator on $S^1 \times \mathbf{R}$, which is a simple case of an operator verifying Hörmander's hypoellipticity theorem [**14**], and at the same time, coincides, up to important lower order terms, with the hypoelliptic Laplacian. Formal conjugation arguments are used to relate the hypoelliptic Laplacian to the elliptic Laplacian on S^1. The fact that the hypoelliptic Laplacian interpolates in the proper sense between the Laplacian and the generator of the geodesic flow can be exhibited by hand. One obtains this way a proof of Poisson's formula by interpolation.

In subsection 1.3, we show that our hypoelliptic Laplacian is a Hodge Laplacian with respect to an exotic bilinear form on the space of compactly supported differential forms on $S^1 \times \mathbf{R}$. This result will be used in section 3 to construct the geometric hypoelliptic Laplacian in the context of de Rham theory.

1.1. Elliptic and hypoelliptic operators. — Let X be a compact manifold. Let $\mathcal{X}^*$ be the total space of T^*X. Then X embeds in $\mathcal{X}^*$ as the zero section of T^*X.

Let E and F be two complex vector bundles on X. If P is a pseudodifferential operator of order m mapping $C^{\infty}(X,E)$ into $C^{\infty}(X,F)$, its principal symbol $\sigma_P(x,\xi)$ is a smooth map on $\mathcal{X}^* \setminus X$ with values in $\mathrm{Hom}(E,F)$, which is homogeneous of order m in the variable ξ. The operator P is said to be elliptic if $\sigma_P(x,\xi)$ is invertible on $\mathcal{X}^* \setminus X$.

If X is equipped with a Riemannian metric, if Δ^X is the Laplace-Beltrami operator acting on $C^{\infty}(X,\mathbf{R})$, then $-\Delta^X$ is an elliptic operator of order 2, and its principal symbol is $|\xi|^2$. The standard example is the operator $-\frac{\partial^2}{\partial x^2}$ acting on S^1.

Ellipticity is a stable property. Indeed a small deformation of an elliptic operator is still elliptic. This should make all the more surprising the fact that certain elliptic operators can be deformed into hypoelliptic operators. This is only possible because the deformed operators act on a different space than the original operator. Besides elliptic operators of order m act on Sobolev spaces, and decrease the Sobolev index by m. As an example, the operator $-\Delta^X$ decreases the Sobolev index by 2, and any pseudoinverse of $-\Delta^X$ (an inverse up to regularizing operators) increases the Sobolev index by 2. In particular if u is a scalar distribution on X such that $-\Delta^X u \in H^s$, then $u \in H^{s+2}$.

Hypoellipticity is a weaker property. A pseudodifferential operator P is said to be hypoelliptic if when u is a distribution such that Pu is C^{∞} on some open set, then u is also C^{∞} on this open set. For example the parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta^X$ on $\mathbf{R} \times X$ is hypoelliptic.

1.2. The Kolmogorov operator and Hörmander's theorem. — Consider the operator A on $\mathbf{R} \times \mathbf{R}^2$ introduced by Kolmogorov [**15**],

$$A = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial x}. \tag{1.1}$$

In [**15**], Kolmogorov computed the fundamental solution of (1.1), as a time dependent Gaussian kernel in the variables (x,y), from which the hypoellipticity of A follows.

The hypoellipticity of A prompted Hörmander [**14**] to develop his theory of hypoelliptic second order differential operators which we now briefly describe. Indeed if $X_0,\dots,X_m$ are smooth vector fields on $\mathbf{R}^n$, consider the differential operator

$$M = -\frac{1}{2}\sum_{i=1}^{m} X_i^2 + X_0. \tag{1.2}$$

Let $\mathcal{E}(x) \subset \mathbf{R}^n$ be the vector space spanned at x by $X_0,\dots,X_m$ and their Lie brackets. Hörmander's theorem asserts that if at each x, $\mathcal{E}(x) = \mathbf{R}^n$, then M is a hypoelliptic operator.