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VESTISLAV APOSTOLOV
DAVID M. J. CALDERBANK
PAUL GAUDUCHON
CHRISTINA W. TØNNESEN-FRIEDMAN

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EXTREMAL KÄHLER METRICS ON RULED MANIFOLDS AND STABILITY

by

Vestislav Apostolov, David M. J. Calderbank, Paul Gauduchon
& Christina W. Tønnesen-Friedman

Abstract. — This article gives a detailed account and a new presentation of a part of our recent work [**3**] in the case of admissible ruled manifolds without blow-downs. It also provides additional results and pieces of information that have been omitted or only sketched in [**3**].

Résumé **(Métriques kähleriennes extrêmes).** — Cet article fournit une étude détaillée et une nouvelle présentations d'une partie de notre travail récent [**3**] dans le cas des variétés admissibles réglées sans *blow-down*. Il fournit également des résultats supplémentaires et des informations qui ont été omis ou simplement esquissés dans [**3**].

Introduction

Compact complex manifolds which admit hamiltonian 2-forms of order 1 in the sense of [**1, 2**]—cf. Section 1.8 for a formal definition—have been classified in [**2**] and extensively studied in [**3**]. The main motivation in [**3**] for studying this class of Kähler manifolds is the fact that they provide a fertile testing ground for the conjectures relating extremal and CSC Kähler metrics to stability. In particular, by using recent results of X. Chen–G. Tian, here quoted as Theorem 2.1, we were able to solve in [**3**] a long pending open question since [**42**], namely the non-existence of extremal Kähler metrics in "large" Kähler classes on "pseudo-Hirzebruch surfaces", which was the last missing step towards the full resolution of the existence problem of extremal Kähler metrics on geometrically ruled complex surfaces [**5**].

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The main goal of this paper is to present some salient results of our joint work [**3**]. To simplify the exposition, we here only consider the simple case of $\mathbb{P}^1$-bundles over a product of compact Kähler manifolds of constant scalar curvature, which in the terminology in [**3**] is referred to as the case *without blow-downs*. This allows us for a specific treatment, somewhat simpler than the general case worked out in [**3**], to which we refer the reader for more information and details.

For the comfort of the reader, we tried to make this paper as self-contained and easy to read as possible. With regard to [**3**], we introduce in places slightly different notation and terminology, that seem to be more adapted to the specific situations worked out in this paper. Similarly, some computations and arguments taken from [**3**] here appear in a slightly different and/or a more detailed presentation. The paper also includes new pieces of information, which were omitted or only sketched in [**3**], like Proposition 1.5 in Section 1.9, Proposition A.1 in Appendix A, a specific account of the deformation to the normal cone of the infinity section in admissible ruled manifolds, etc.

The paper is organized as follows.

In Sections 1.1 to 1.7, we set the general framework of the paper by introducing the class of *admissible ruled manifolds*, the cone of *admissible Kähler classes*, the set of *admissible momenta* and the associated set of of *admissible Kähler metrics*, and by recalling the main geometric features of these metrics (isometry groups, Ricci form, scalar curvature, etc.). In Section 1.8, we briefly explain how hamiltonian 2-forms of order 1 arise in this setting. In Section 1.9, we use a variant of the Calabi method in [**8**], also used in [**42**], to construct extremal admissible Kähler metrics in a given admissible Kähler class Ω; as in [**42**], we show that this method works successfully if and only if the *extremal polynomial* F_Ω, canonically attached to Ω, is positive on its interval of definition. Section 1.10 is devoted to the special case of admissible ruled *surfaces*, here called *Hirzebruch-like ruled surfaces.*

In Section 2.1, we review some well-known general facts concerning the space of Kähler metrics in a given Kähler class on a compact complex manifold. In Section 2.2, we recall some basic results recently obtained by X. X. Chen and G. Tian, here stated as Theorem 2.1, which play an important role in several parts of the paper. In Section 2.3, we compute the relative Mabuchi K-energy on the space of admissible Kähler metrics in any admissible Kähler class Ω and we show that Ω admits an extremal Kähler metric, which is then admissible up to automorphism, if and only if F_Ω is positive on its interval of definition (Theorem 2.2). Proposition A.1 established in Appendix A is used to complete the proof of Theorem 2.2 in the *borderline case*, when F_Ω is non-negative but has zeros, possibly irrational, in its interval of definition.

In Section 3.1, we recall the interpretation given by Donaldson and adapted by Székelyhidi to the relative case of the Futaki invariant of an S^1-action on a general polarized projective manifold. In Section 3.2, we construct the *deformation to the normal cone*, $\mathcal{D}(M)$, of the infinity section Σ_∞ of an admissible ruled manifold M. In Section 3.3, for any admissible polarization Ω on M, we turn $\mathcal{D}(M)$ into a *test configuration* in the sense of Tian [**41**] and Donaldson [**15**], by constructing a family of relative polarizations, parametrized by rational numbers in the interval of definition of the extremal polynomial F_Ω. In Section 3.4, we extend to admissible ruled manifolds a beautiful computation done by G. Székelyhidi [**39**] for ruled surfaces, and we show that, for any rational number x in $(-1,1)$, $F_\Omega(x)$ is equal, up to a constant (negative) factor, to the relative Futaki invariant of the test configuration $\mathcal{D}(M)$ equipped with the relative polarization determined by x, see Theorem 3.1. Together with Theorem 2.2, this striking—and still mysterious—fact has the following consequence: for admissible ruled manifolds and admissible Kähler classes, the relative slope K-stability, as defined by J. Ross and R. Thomas [**35, 34**], implies the existence of extremal Kähler metrics, cf. [**3**, Theorem 2]. For a more detailed discussion on this matter, including the role of the examples of Section 2.4 in the current refined definitions of the slope stability, the reader is referred to [**3**, Theorem 2].

Notation and convention. — For any Kähler structure (g, J, ω), the riemannian metric g, the complex structure J and the Kähler form ω are linked together by $\omega = g(J\cdot, \cdot)$. The Levi-Civita connection of g, as a covariant derivative acting on any sorts of tensor fields, will be denoted by D^g, or simply D when the metric is understood. The twisted differential d^c acting on exterior forms is defined by $d^c = JdJ^{-1}$, where J acts on a p-form φ by $(J\varphi)(X_1, \ldots, X_p) = \varphi(J^{-1}X_1, \ldots, J^{-1}X_p)$; in terms of the operators ∂ and $\bar\partial$ we then have $d^c = i(\bar\partial - \partial)$ and $dd^c = 2i\partial\bar\partial$. Our overall convention for the curvature of a linear connection ∇ is $R^\nabla_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

1. Extremal metrics on admissible ruled manifolds

1.1. Admissible ruled manifolds. — Unless otherwise specified, M will denote a connected, compact, complex manifold of complex dimension $m \geq 2$, of the form

$$M = \mathbb{P}(1 \oplus L), \tag{1.1}$$

where L denotes a holomorphic line bundle over some (connected, compact) complex manifold S of complex dimension $(m-1)$. Here, 1 stands for the trivial holomorphic line bundle $S \times \mathbb{C}$ and $\mathbb{P}(1 \oplus L)$ then denotes the projective line bundle associated to the holomorphic rank 2 vector bundle $E = 1 \oplus L$: an element ξ of M over a point y of S is then a complex line through the origin in the complex 2-plane $E_y = \mathbb{C} \oplus L_y$, where E_y, L_y denote the fibres of E, L at y; if ξ is generated by the pair (z, u) in

$\mathbb{C} \oplus L_y$, we write $\xi = (z : u)$. The natural (holomorphic) projection $\pi : M \to S$ admits two natural (holomorphic) sections: the *zero section* $\sigma_0 : y \mapsto \mathbb{C} \subset \mathbb{C} \oplus L_y$, and the *infinity section* $\sigma_\infty : y \mapsto L_y \subset \mathbb{C} \oplus L_y$. We denote by Σ_0, Σ_∞ the images of σ_0, σ_∞ in M, still called zero section and infinity section, both identified with S via π. Each element of $M \setminus \Sigma_\infty$ over y has a unique generator of the form $(1, u)$, with u in L_y: we thus get a natural identification of $M \setminus \Sigma_\infty$ with L and M can therefore be regarded as a compactification of (the total space of) L obtained by adding a *point at infinity* to each fiber. The open set $M_0 = M \setminus (\Sigma_0 \cup \Sigma_\infty)$ is similarly identified with the set of non-zero elements of L.

The natural $\mathbb{C}^*$-action on L extends to a holomorphic $\mathbb{C}^*$-action on M defined by: $\zeta \cdot (z : u) = (z : \zeta\, u)$. This action pointwise fixes Σ_0 and Σ_∞. The vector field on M generating the induced S^1-action is denoted by T.

We furthermore assume that $S = \prod_{i=1}^N S_i$ is the product of $N \geq 1$ (connected, compact) complex manifolds S_i, of complex dimensions d_i, and that L comes equipped with a (fiberwise) hermitian inner product, h, such that the curvature, R^∇, of the corresponding Chern connection, ∇, is of the form: $R^\nabla = -i \sum_{i=1}^N \epsilon_i\, \omega_{S_i}$, where each ω_{S_i} is the Kähler form of a Kähler metric, g_{S_i}, on S_i (viewed as a 2-form on S. i.e. identified with $p_i^* \omega_i$, if p_i denotes the natural projection from S to S_i), and ϵ_i is equal to 1 or to -1. In particular, $\sum_{i=1}^N \epsilon_i\, [\omega_{S_i}] = 2\pi\, c_1(L^*)$, where $c_1(L^*)$ denotes the first Chern class of the dual complex line bundle L^* and $[\omega_{S_i}]$ the class of ω_{S_i} in $H^2(S, \mathbb{R})$.

Moreover, for $i = 1, \ldots, N$, we assume that $R^{\nabla_i} = -i\epsilon_i \omega_{S_i}$ is the Chern curvature of a hermitian holomorphic line bundle, L_i, on S_i—so that (S_i, ω_{S_i}) is polarized by $\tilde{L}_i = L_i^{-\epsilon_i}$—and that $L = \otimes_{i=1}^N p_i^* L_i$, equipped with the induced (fiberwise) hermitian metric.

On M_0, identified with $L \setminus \Sigma_0$ as above, define t by

$$t = \log r, \tag{1.2}$$

where $r = |\cdot|_h$ denotes the norm relative to h, viewed as a function on $L = M \setminus \Sigma_\infty$. We then have

$$d^c t(T) = 1, \qquad dd^c t = \pi^*(\sum_{i=1}^N \epsilon_i\, \omega_{S_i}), \tag{1.3}$$

where the twisted differential operator d^c, as defined above, is relative to the natural complex structure of M. The latter, as well as the complex structures of S and of each factor S_i, will be uniformly denoted by J and will be kept unchanged throughout the paper.

Definition 1.1. — Ruled manifolds of the above kind, with the additional pieces of structure described in this section, will be referred to as *admissible ruled manifolds.*