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ON CERTAIN UNITARY GROUP SHIMURA VARIETIES

by

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Abstract. — In this paper, we study the local geometry at a prime p of a certain class of (PEL) type Shimura varieties. We begin by studying the Newton polygon stratification of the special fiber of a Shimura variety with good reduction at p. Each stratum can be described in terms of the products of the reduced fiber of the corresponding Rapoport-Zink space with some smooth varieties (we call the Igusa varieties), and of the action on them of a certain p-adic group T_α, which depends on the stratum. (The definition of the Igusa varieties in this context is based upon a result of Zink on the slope filtration of a Barsotti-Tate group and on the notion of Oort's foliation.) In particular, we show that it is possible to compute the étale cohomology with compact supports of the Newton polygon strata, in terms of the étale cohomology with compact supports of the Igusa varieties and the Rapoport-Zink spaces, and of the group homology of T_α. Further more, we are able to extend Zariski locally the above constructions to characteristic zero and obtain an analogous description for the étale cohomology of the Shimura varieties in both the cases of good and bad reduction at p. As a result of this analysis, we obtain a description of the ℓ-adic cohomology of the Shimura varieties, in terms of the ℓ-adic cohomology with compact supports of the Igusa varieties and of the Rapoport-Zink spaces.

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***Résumé* (Sur certaines variétés de Shimura associées à des groupes unitaires)**
Dans cet article, nous étudions la géométrie locale, en un nombre premier p, d'une certaine classe de variétés de Shimura de type PEL. Nous commençons par étudier la stratification par le polygone de Newton de la fibre spéciale des variétés de Shimura ayant bonne réduction en p. Chaque strate peut être décrite en termes de produits des fibres réduites des espaces de Rapoport-Zink correspondants avec certaines variétés lisses, les variétés d'Igusa, et de l'action sur ces objets d'un certain groupe p-adique T_α, qui dépend de la strate. Nous montrons en particulier qu'il est possible de calculer la cohomologie étale à support compact des strates du polygone de Newton, en termes de la cohomologie étale à support compact des variétés d'Igusa et des espaces de Rapoport-Zink, et de l'homologie des groupes de T_α. De plus, nous parvenons à étendre localement (au sens de la topologie de Zariski) les constructions précédentes à la caractéristique nulle et à obtenir une description analogue de la cohomologie étale des variétés de Shimura, dans les cas de bonne comme de mauvaise réduction en p. Comme conséquence de cette étude, nous obtenons une description de la cohomologie ℓ-adique des variétés de Shimura, en termes de la cohomologie ℓ-adique à support compact des variétés d'Igusa et des espaces de Rapoport-Zink.

1. Introduction

In this paper, we study a certain class of (PEL) type Shimura varieties. These varieties arise as moduli spaces of polarized abelian varieties, endowed with an action of a division algebra and a level structure. Their ℓ-adic cohomology is the object of a conjecture of Langlands.

In [**27**] Rapoport and Zink introduce local analogues of the Shimura varieties, which are (PEL) type moduli spaces for Barsotti-Tate groups, in the category of rigid analytic spaces. These spaces can be used to give rigid analytic uniformizations of isogeny classes of abelian varieties inside the corresponding Shimura varieties. In [**26**] Rapoport reports a conjecture of Kottwitz for the ℓ-adic cohomology groups with compact supports of the Rapoport-Zink spaces. This conjecture is "heuristically compatible" (in the sense of the p-adic uniformization given in [**27**]) with the corresponding global conjecture on Shimura varieties.

In [**14**] Harris and Taylor prove the local Langlands conjecture by studying a particular class of (PEL) type Shimura varieties. In their work, they analyse the reduction mod p of the Shimura varieties via the notion of Igusa varieties. These varieties arise as finite étale covers of the locus, inside the reduction of the Shimura varieties with no level structure at p, where the Barsotti-Tate group associated to the abelian variety lies in a fixed isomorphism class. Their analysis strongly relies on the fact that, for the class of Shimura varieties they consider, the pertinent Barsotti-Tate groups are one dimensional, and thus Drinfeld's theory of elliptic modules applies.

For general (PEL) type Shimura varieties such an assumption on the dimension of the Barsotti-Tate groups which control the deformation of the abelian varieties does not hold. On the other hand, it might be possible to describe the geometry and the cohomology of general (PEL) type Shimura varieties by combining together Harris-Taylor's and Rapoport-Zink's techniques. We consider the Newton polygon stratification of the reduction of the Shimura varieties, which is defined by the loci where the Barsotti-Tate group associated to the abelian variety lies in a fixed isogeny class. The idea is to analyse each Newton polygon stratum along two main "directions": one corresponding to deforming the abelian varieties without altering the isomorphism class of the associated Barsotti-Tate group, the other corresponding to varying the abelian varieties inside one isogeny class.

In this paper, we carry out this plan for a simple class of (PEL) type Shimura varieties and, as a result, we obtain a description of the ℓ-adic cohomology groups of the Shimura varieties, in terms of the ℓ-adic cohomology with compact supports groups of the Igusa varieties and of the Rapoport-Zink spaces, in the appropriate Grothendieck group, for any prime number $\ell \neq p$.

More precisely, the class of (PEL) type Shimura varieties we are interested in arises as the class of moduli spaces of polarized abelian varieties endowed with the action of a division algebra and with a level structure associated to the data $(E, B, *, V, \langle \bullet, \bullet \rangle)$ where:

- E is an imaginary quadratic extension of $\mathbb{Q}$ in which the prime p splits (we write $(p) = u \cdot u^c$);
- B is a central division algebra over E of dimension h^2 which splits at u;
- $*$ is a positive involution of the second kind on B;
- $V = B$ viewed as a B-module;
- $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{Q}$ is a non degenerate alternating $*$-hermitian pairing.

We denote by G the algebraic group over $\mathbb{Q}$ of the automorphisms of V which preserves $\langle \bullet, \bullet \rangle$ up to scalar multiple, and by G_1 the algebraic subgroup of G of the automorphisms which preserves $\langle \bullet, \bullet \rangle$, *i.e.* $0 \to G_1 \to G \to \mathbb{G}_m \to 0$. Finally, we also assume

$$G_1(\mathbb{R}) = U(q, h-q),$$

for some integer q, $1 \leqslant q \leqslant h-1$.

We remark that when $q = 1$ the above class of Shimura varieties is a subclass of the one studied by Harris and Taylor in [**14**] (namely, the case when the totally real part of the ground field is trivial).

For any sufficiently small open compact subgroup $U \subset G(\mathbb{A}^\infty)$, we call the Shimura variety of level U the smooth projective scheme X_U over Spec E, of dimension $q(h-q)$, which arises as the moduli space of polarized abelian varieties endowed with a *compatible* action of B and with a structure of level U, classified up to isogeny (see [**22**]).

Our goal is to study the virtual representation of the group $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$:

$$H(X, \mathbb{Q}_\ell) = \sum_{i \geqslant 0} (-1)^i \varinjlim_U H^i_{\text{ét}}(X_U \times_E (\widehat{E}^{\text{nr}}_u)^{\text{ac}}, \mathbb{Q}_\ell).$$

We obtain the following theorem.

***Theorem 1* (Main Theorem).** — *There is an equality of virtual representations of the group $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$:*

$$H(X,\mathbb{Q}_\ell)^{\mathbb{Z}_p^\times} = \sum_{\alpha,k,i,j} (-1)^{k+i+j} \varinjlim_{V_p} \mathrm{Ext}^k_{T_\alpha\text{-smooth}}\Big(H^i_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,V_p},\mathbb{Q}_\ell(D)), H^j_c(J_\alpha,\mathbb{Q}_\ell)\Big)$$

where:

- *$D = q(h-q)$ is the dimension of the Shimura varieties;*
- *the action of $\mathbb{Z}_p^\times$ on $H(X,\mathbb{Z}/\ell^r\mathbb{Z})$ is defined via the embedding*

$$\mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times \times (B_u^{\mathrm{op}})^\times = G(\mathbb{Q}_p) \subset G(\mathbb{A}^\infty);$$

- *α varies among all the Newton polygons of height h and dimension q;*
- *for each α, T_α is a p-adic group of the form $T_\alpha = \prod_i \mathrm{GL}_{r_i}(D_i)$ for some finite dimensional division algebras $D_i/\mathbb{Q}_p$;*
- *$H^j_c(J_\alpha,\mathbb{Q}_\ell)$ are representations of $T_\alpha \times G(\mathbb{A}^{\infty,p}) \times (\mathbb{Q}_p^\times/\mathbb{Z}_p^\times) \times (W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p})$ associated to the ℓ-adic cohomology with compact support groups of the Igusa varieties, for all $j \geqslant 0$;*
- *$H^k_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,V_p},\mathbb{Q}_\ell)$ are the ℓ-adic cohomology with compact supports groups of the rigid analytic Rapoport-Zink space of level V_p, for all $k \geqslant 0$ and any open compact subgroup $V_p \subset G(\mathbb{Q}_p)/\mathbb{Q}_p^\times$; as the level V_p varies, they form a direct limit of representations of $T_\alpha \times W_{\mathbb{Q}_p}$, endowed with an action of $G(\mathbb{Q}_p)/\mathbb{Q}_p^\times$.*

In the following, we outline in more detail the content of this paper.

In [**22**] Kottwitz proves that the Shimura varieties without level structure at p, *i.e.* associated to a subgroup U of the form

$$U = U^p(0) = U^p \times \mathbb{Z}_p^\times \times \mathcal{O}^\times_{B_u^{\mathrm{op}}},$$

admit smooth integral models over $\operatorname{Spec}\mathcal{O}_{E_u}$ ($\mathcal{O}_{E_u} = \mathbb{Z}_p$). We denote by $\overline{X} = \overline{X}_{U^p(0)}$ the reduction $X_{U^p(0)} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$ of a Shimura variety with no level structure at p and by $\mathcal{A}$ the universal abelian variety over $\overline{X}$. It follows from the definition of the moduli space and Serre-Tate's theorem that the deformation theory of the abelian variety $\mathcal{A}$ over $\overline{X}$ is controlled by a Barsotti-Tate group of height h and dimension q over $\overline{X}$, which we denote by $\mathcal{G}/\overline{X}$ ($\mathcal{G} \subset \mathcal{A}[p^\infty]$).

In [**24**] Oort studies the Newton polygon stratification of a moduli space of abelian varieties in positive characteristic. This is a stratification by locally closed subschemes which are defined in terms of the Newton polygons of the Barsotti-Tate groups associated to the abelian varieties. (Newton polygons associated to Barsotti-Tate groups were first introduced and studied by Grothendieck in [**13**] and Katz in [**19**]). For any Newton polygon α of dimension q and height h, the associated stratum $\overline{X}^{(\alpha)}$ is the locus where the Barsotti-Tate group $\mathcal{G}$ has constant Newton polygon equal to α, *i.e.* constant isogeny class.

The first step towards the main theorem is to notice that the decomposition of $\overline{X}$ as the disjoint union of the open Newton polygon strata $\overline{X}^{(\alpha)}$ induces an equality of virtual representations of the group $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$:

$$\sum_{i\geqslant 0}(-1)^i H^i(X,\mathbb{Q}_\ell) = \sum_{i\geqslant 0}(-1)^i H^i(\overline{X},\mathbb{Q}_\ell) = \sum_\alpha \sum_{j\geqslant 0}(-1)^j H^j_c(\overline{X}^{(\alpha)},\mathbb{Q}_\ell).$$

Thus, we may restrict ourself to study each Newton polygon stratum separately. Since we are assuming $q \geqslant 1$, to each isogeny class of Barsotti-Tate groups of dimension q and height h correspond possibly many distinct isomorphism classes. It is a result of Oort that, for any given Barsotti-Tate group H over $\overline{\mathbb{F}}_p$, with Newton polygon equal to α, the set of geometric points x of $\overline{X}^{(\alpha)}$ such that $\mathcal{G}_x \simeq H$ is a closed subset of $\overline{X}^{(\alpha)} \times \overline{\mathbb{F}}_p$. Moreover, the corresponding reduced subscheme of $\overline{X}^{(\alpha)} \times \overline{\mathbb{F}}_p$ is a smooth scheme over $\operatorname{Spec} \overline{\mathbb{F}}_p$, which is called the leaf associated to H and is denoted by C_H.

In this work, we focus our attention on a distinguished leaf $C_\alpha = C_{\Sigma_\alpha}$ inside each Newton polygon stratum, which we call the central leaf, and define the Igusa varieties as covering spaces of the central leaf C_α. Before introducing the definition of Igusa variety, we recall a result of Zink (see [**29**]). This result extends the classical result in Dieudonné's theory of p-divisible groups which states that any p-divisible group defined over a perfect field is isogenous to a completely slope divisible one, *i.e.* to a direct product of isoclinic slope divisible p-divisible groups. In [**29**] Zink shows that over a regular scheme of characteristic p any p-divisible group with constant Newton polygon of slopes $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ is isogenous to a p-divisible group $\mathcal{G}$ which admits a filtration (called the slope filtration)

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G}$$

whose factors $\mathcal{G}^i = \mathcal{G}_i/\mathcal{G}_{i-1}$ are isoclinic slope divisible p-divisible groups of slope λ_i. In particular, it follows from Zink's work that the Barsotti-Tate group $\mathcal{G}$ over C_α admits a slope filtration (see remark 2.14).

Definition 2. — For any positive integer m, we define the Igusa variety of level m, $J_{\alpha,m}$, over C_α to be the universal space for the existence of isomorphisms

$$j^i_m : \Sigma^i[p^m] \longrightarrow \mathcal{G}^i[p^m]$$

which extend étale locally to any higher level $m' \geqslant m$ (we denote by Σ^i the isoclinic piece of Σ_α of slope λ_i, for each i).

The notion of Igusa varieties was first introduced by Igusa in [**16**] in the theory of elliptic curves and used to describe the reduction at a bad prime p of modular curves (see [**20**]). In [**14**], Harris and Taylor introduce and study a higher dimensional analogue of the Igusa curves which they use to describe the reduction at a bad prime p of Shimura varieties. We remark that, both in the classical theory of modular curves and in the case of the Shimura varieties considered by Harris and Taylor in [**14**], the Igusa varieties are finite étale covers of the whole open Newton polygon stratum, and not of the central leaf. This is because, in the case when $\mathcal{G}$ is one dimensional, there is a unique leaf inside each open Newton polygon stratum, namely the whole stratum