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THE MUMFORD CONJECTURE
[after Madsen and Weiss]

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1. INTRODUCTION

The Mumford conjecture concerns the cohomology of the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g: Mumford constructed tautological classes κ_i, for $i \geqslant 1$, in the Chow ring $\mathrm{CH}^i(\mathcal{M}_g)$ with rational coefficients, which yield a natural morphism of algebras $\mathbb{Q}[\kappa_i] \to \mathrm{CH}^*(\mathcal{M})$, in which $\mathrm{CH}^*(\mathcal{M})$ denotes the Chow ring of the moduli spaces, stabilized with respect to the genus. The conjecture asserts that the above morphism is an isomorphism [19, 8].

The conjecture can be reformulated in terms of the stable cohomology of the mapping class groups (or Teichmüller modular groups) Γ_g [5, 17]. The mapping class group Γ_g is the discrete group of isotopy classes of orientation-preserving diffeomorphisms of a smooth, oriented surface of genus g. The group cohomology $H^*(B\Gamma_g)$ of the mapping class groups stabilizes in a given degree for sufficiently large genus. The stable value identifies with the cohomology of the space $B\Gamma_\infty$, which is the homotopy colimit of the system of classifying spaces $B\Gamma_{g,2}$ of the mapping class groups of curves with two marked points, stabilized with respect to maps induced by group morphisms $\Gamma_{g,2} \to \Gamma_{g+1,2}$.

The moduli space $\mathcal{M}_g$ can be constructed, as an analytic space, as the quotient of the action of the group Γ_g upon Teichmüller space, $\mathcal{T}_g$. Teichmüller space is contractible and the action has finite isotopy groups, hence the Mumford conjecture can be restated in terms of the Mumford-Morita-Miller characteristic classes [14, 16], $\kappa_i \in H^{2i}(B\Gamma_\infty;\mathbb{Q})$.

Conjecture 1.1. — *The classes $\kappa_i \in H^{2i}(B\Gamma_\infty;\mathbb{Q})$ induce an isomorphism of algebras $\widetilde{\alpha} : \mathbb{Q}[\kappa_i] \to H^*(B\Gamma_\infty;\mathbb{Q})$.*

The algebra $H^*(B\Gamma_\infty;\mathbb{Q})$ has a Hopf algebra structure, induced by a multiplicative structure of geometric origin on the classifying space $B\Gamma_\infty$. The classes κ_i are primitive and non-trivial, thus the morphism $\widetilde{\alpha}$ is a monomorphism of Hopf algebras.

The space $B\Gamma_\infty$ has a structure which enriches the multiplicative structure; namely, the space $B\Gamma_\infty$ has a perfect fundamental group, hence the Quillen plus construction applies to yield a morphism $B\Gamma_\infty \to B\Gamma_\infty^+$, which induces an isomorphism in homology and such that $B\Gamma_\infty^+$ has trivial fundamental group. Tillmann [24] showed that the space $\mathbb{Z}\times B\Gamma_\infty^+$ is an infinite loop space, hence it represents the degree zero part of a generalized cohomology theory; the identification of the associated cohomology theory is a problem of *stable* homotopy theory.

The construction of the Mumford-Morita-Miller characteristic classes uses integration along the fibre of powers of the first Chern class of the orientation bundle of the universal oriented surface bundle. This can be interpreted in terms of the Gysin morphism, which is of topological origin, via the Pontrjagin-Thom construction. Madsen and Tillmann [11] constructed a morphism of infinite loop spaces

$$\alpha_\infty : \mathbb{Z}\times B\Gamma_\infty^+ \longrightarrow \Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$$

which lifts the construction of $\widetilde{\alpha}$. The infinite loop space $\Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ is constructed from the Thom spectrum which is associated to the complements of the canonical line bundles on complex projective space.

The rational cohomology of the space $\Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ is isomorphic to the rational cohomology of the space $\mathbb{Z}\times BU$, where BU denotes the classifying space of the infinite unitary group. The cohomology algebra $H^*(BU;\mathbb{Q})$ is isomorphic to the polynomial algebra $\mathbb{Q}[\kappa_i]$, where the classes κ_i can be taken to be Chern classes, hence the Mumford conjecture is implied by the following result, which is referred to as the generalized Mumford conjecture.

Theorem 1.2 ([12]). — *The morphism* $\alpha_\infty : \mathbb{Z}\times B\Gamma_\infty^+ \to \Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ *is a homotopy equivalence.*

The cohomology of the space $\Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ with coefficients in a finite field $\mathbb{F}_p$ has been calculated [4], using techniques of algebraic topology. The above theorem therefore yields a calculation of the stable cohomology of the mapping class groups $H^*(B\Gamma_\infty;\mathbb{F}_p)$, for any prime p.

1.1. Methods of proof

Madsen and Weiss reformulate the generalized Mumford conjecture using certain generalized bundle theories; these are local in nature and their classifying spaces can be constructed from realization spaces associated to sheaves of sets. In particular, they give an interpretation of a modification of the morphism α_∞ introduced in [11] as the realization of a morphism of sheaves.

Let $\mathfrak{X}$ denote the category of smooth manifolds, without boundary and with a countable basis and consider sheaves of sets on $\mathfrak{X}$. There is a natural notion of homotopy on the sections of a sheaf, termed *concordance*; if $\mathcal{F}$ is a sheaf and X is a smooth manifold, then concordance is an equivalence relation on the sections $\mathcal{F}(X)$, which is induced by elements of $\mathcal{F}(X \times \mathbb{R})$, in the usual way. The set of equivalence classes for the concordance relation is written as $\mathcal{F}[X]$. The contravariant functor $X \mapsto \mathcal{F}[X]$ is represented on the homotopy category of topological spaces by a space $|\mathcal{F}|$; namely $\mathcal{F}[X] \cong [X, |\mathcal{F}|]$, where the right hand side denotes homotopy classes of morphisms of topological spaces. A morphism $f : \mathcal{E} \to \mathcal{F}$ of sheaves is defined to be a weak equivalence if the induced morphism $|f| : |\mathcal{E}| \to |\mathcal{F}|$ is a weak equivalence(1) between the representing spaces.

There are two principal techniques which are used to show that a morphism between sheaves is a weak equivalence: to exhibit explicit concordances so as to obtain an isomorphism of concordance classes or to use the relative surjectivity criterion of Proposition A.7 to show that a morphism is a weak equivalence.

The classifying space $B\Gamma_g$ classifies bundles with fibres which are closed oriented surfaces, hence the source of the morphism α_∞ is related to bundles of closed oriented surfaces. This motivates consideration of the sheaf $\mathcal{V}$ with sections over X the set of pairs (π, f), where $\pi : E \to X$ is a smooth submersion with 3-dimensional oriented fibres and $f : E \to \mathbb{R}$ is a smooth morphism such that (π, f) is a proper submersion. Ehresmann's fibration lemma implies that this is a bundle of smooth surfaces on $X \times \mathbb{R}$.

The definition of $\mathcal{V}$ can be weakened: let $h\mathcal{V}$ denote the sheaf with sections over X the set of pairs $(\pi, \widehat{f})$, where $\pi : E \to X$ is as before and $\widehat{f}$ is a smooth section of the fibrewise 1-jet bundle $J^1_\pi(E, \mathbb{R}) \to E$, subject to the condition that the morphism $(\pi, f) : E \to X \times \mathbb{R}$ is a proper submersion, where f denotes the underlying smooth map, $f : E \to \mathbb{R}$, of $\widehat{f}$. There is a morphism of sheaves $\alpha : \mathcal{V} \to h\mathcal{V}$, induced by jet prolongation, which induces a morphism of topological spaces $|\alpha| : |\mathcal{V}| \to |h\mathcal{V}|$, which is related to the morphism α_∞.

These definitions generalize; namely it is expedient to allow mild fibrewise singularities over $X \times \mathbb{R}$, by considering smooth sections of the fibrewise 2-jet bundle $J^2_\pi(E, \mathbb{R}) \to E$ and permitting fibrewise critical points which are of Morse type. This gives sheaves $\mathcal{W}, h\mathcal{W}$, where $\mathcal{W}$ corresponds to the integrable situation, as above. Similarly, there are sheaves $\mathcal{W}_{\text{loc}}, h\mathcal{W}_{\text{loc}}$ which correspond to the local situation around the singular sets and these sheaves form a commutative diagram

$$\begin{array}{ccccc} \mathcal{V} & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}_{\text{loc}} \\ {\scriptstyle j^2_\pi}\downarrow & & {\scriptstyle j^2_\pi}\downarrow & & \downarrow{\scriptstyle j^2_\pi} \\ h\mathcal{V} & \longrightarrow & h\mathcal{W} & \longrightarrow & h\mathcal{W}_{\text{loc}}. \end{array} \tag{1}$$

(1) (induces an isomorphism on homotopy groups)

The first main theorem of Vassiliev on the space of functions with moderate singularities is used to show the following Theorem, which motivates the strategy of proof.

Theorem 1.3. — *The morphism* $j^2_\pi : \mathcal{W} \to h\mathcal{W}$ *is a weak equivalence.*

This result is used in conjunction with the following, which is proved using bordism theory.

Theorem 1.4

(1) *The morphism* $j^2_\pi : \mathcal{W}_{\mathrm{loc}} \to h\mathcal{W}_{\mathrm{loc}}$ *is a weak equivalence.*

(2) *The sequence of representing spaces* $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$ *is a homotopy fibre sequence*[(2)] *of infinite loop spaces.*

(3) *There is a homotopy equivalence* $|h\mathcal{V}| \simeq \Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-1})$.

Let F denote the homotopy fibre of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$, then it follows formally from the homotopy invariance of the homotopy fibre construction that there is a homotopy equivalence $F \xrightarrow{\simeq} |h\mathcal{V}|$. Standard methods of homotopy theory imply that the generalized Mumford conjecture follows from:

Theorem 1.5. — *There exists a morphism* $\mathbb{Z} \times B\Gamma_\infty \to F$ *which induces an isomorphism in homology with integral coefficients.*

The proof of this theorem involves replacing the singularities inherent in $\mathcal{W}$ by ones in standard form and then stratifying by critical sheets; after stratification, the concordance relation is imposed by a homotopical gluing construction, the homotopy colimit over a suitable category. The proof of the theorem relies on foundational results from homotopy theory together with the homological stability results of Harer; in particular, the proof uses closed surfaces with boundary.

1.2. Approximations

Much of the material of [12] is developed for bundles of manifolds of arbitrary dimension, d, and with a general notion of orientation, the Θ-orientation. For the presentation of this text, the general notion of orientation has been suppressed and the integer d is usually taken to be two.

To avoid set-theoretic difficulties, [12] uses the notion of graphic morphisms with respect to a fixed set in the definitions of the sheaves which are considered; moreover set-theoretic caveats are required in various proofs. All such details have been suppressed in this text.

(2) A sequence of pointed spaces $F \to E \to B$ is a homotopy fibre sequence if F is weakly equivalent to the homotopy fibre of $E \to B$. The homotopy fibre can be defined explicitly as the fibre product $E \times_B PB$, where $PB \to B$ is the path space fibration over B.

2. MAPPING CLASS GROUPS

2.1. Orientation-preserving diffeomorphisms

Let F be a smooth, compact, oriented surface with boundary ∂F, then F is classified, up to diffeomorphism, by its genus g and the number b of boundary components; write $F_{g,b}$ for a representative of the diffeomorphism class.

The topological group of orientation-preserving diffeomorphisms of F which fix the boundary is written $\mathrm{Diff}^{\circ}(F;\partial F)$ and $\mathrm{Diff}^{\circ}{}_e(F;\partial F)$ denotes the connected component which contains the identity, so that there is a canonical monomorphism of topological groups, $\mathrm{Diff}^{\circ}{}_e(F;\partial F) \to \mathrm{Diff}^{\circ}(F;\partial F)$.

Definition 2.1. — *For g, b non-negative integers, the mapping class group $\Gamma_{g,b}$ is the discrete group of path components, $\Gamma_{g,b} := \pi_0(\mathrm{Diff}^{\circ}(F_{g,b};\partial F_{g,b}))$.*

Earle and Eells [2] proved that the topological group $\mathrm{Diff}^{\circ}{}_e(F;\partial F)$ is contractible, for F a smooth, compact, oriented surface of genus $g \geqslant 2$.

Corollary 2.2. — *For $g \geqslant 2$ an integer, there is a homotopy equivalence $B\Gamma_{g,b} \simeq B\mathrm{Diff}^{\circ}(F_{g,b};\partial F_{g,b})$. In particular, the classifying space $B\Gamma_{g,b}$ classifies isomorphism classes of oriented $F_{g,b}$-bundles.*

There is a model for the classifying space $B\Gamma_{g,b}$ constructed from Teichmüller space, for strictly positive b. Let $\mathcal{H}(F)$ denote the space of hyperbolic metrics on the surface F with geodesic boundary such that each boundary circle has unit length. The hyperbolic model for the moduli space of Riemann surfaces of topological type F is given by

$$\mathcal{M}(F) := \mathcal{H}(F)/\mathrm{Diff}^{\circ}(F;\partial F).$$

Teichmüller space is defined as the quotient $\mathcal{T}(F) := \mathcal{H}(F)/\mathrm{Diff}^{\circ}{}_e(F;\partial F)$.

Theorem 2.3 ([2, 3]). — *Let $F := F_{g,b}$ be a smooth, compact, oriented surface of genus $g > 1$, with b boundary components. The following statements hold.*

(1) *The space $\mathcal{H}(F)$ is contractible.*

(2) *The space $\mathcal{T}(F)$ is contractible and homeomorphic to $\mathbb{R}^{6g-6+2b}$.*

(3) *If $b > 0$, the action of $\Gamma_{g,b}$ on Teichmüller space $\mathcal{T}(F)$ is free and $B\Gamma_{g,b} \simeq \mathcal{M}(F)$.*

(4) *If $b = 0$, the action of Γ_g on Teichmüller space $\mathcal{T}(F_{g,0})$ has finite isotropy groups, hence there is a rational homotopy equivalence $B\Gamma_g \simeq_{\mathbb{Q}} \mathcal{M}(F_{g,0})$.*

In particular, the above establishes the relation between the moduli space of Riemann surfaces $\mathcal{M}(F)$ and the mapping class group.