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ON THE LINEARIZATION THEOREM FOR PROPER LIE GROUPOIDS

BY Marius CRAINIC AND Ivan STRUCHINER

ABSTRACT. – We revisit the linearization theorems for proper Lie groupoids around general orbits (statements and proofs). In the fixed point case (known as Zung's theorem) we give a shorter and more geometric proof, based on a Moser deformation argument. The passage to general orbits (Weinstein) is given a more conceptual interpretation: as a manifestation of Morita invariance. We also clarify the precise statements of the Linearization Theorem (there has been some confusion on this, which has propagated throughout the existing literature).

RÉSUMÉ. – Nous revisitons les théorèmes de linéarisation pour les groupoïdes de Lie propres autour des orbites générales. Dans le cas du point fixe (connu sous le nom de théorème de Zung), nous donnons une preuve plus courte et plus géométrique, basée sur l'argument de déformation de Moser. Le passage au cas général est décrit de façon plus conceptuelle, comme manifestation de l'invariance de Morita. Nous clarifions également l'énoncé précis du théorème de linéarisation (la littérature sur ce sujet est assez confuse).

Introduction

The linearization theorem for Lie groupoids is a far reaching generalization of the tube theorem (for Lie group actions), Ehresmann's theorem (for proper submersions), and Reeb stability (for foliations). It was first addressed by A. Weinstein as part of his program that aims at a geometric understanding of Conn's linearization theorem in Poisson Geometry [17]. Various partial results have been obtained over the last 10 years (see Section 1.4). However, even though it was a general belief that *every proper Lie groupoid is linearizable*, such a statement has never been made precise or proven. Moreover, even for the existing results there has been some confusion regarding their precise statements (see also Section 1.4 below); this confusion has propagated throughout the existing literature.

The aim of this note is to clarify the statement of the linearization theorem and to present a simple geometric proof of it. Recall that, for any orbit $\mathcal{O}$ of a Lie groupoid $\mathcal{G}$, there is a local

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model for $\mathcal{G}$ around $\mathcal{O}$, the linearization $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$. The theorem refers to the equivalence of $\mathcal{G}$ and $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ near $\mathcal{O}$. The main theorem we discuss is the following.

THEOREM 1. – *Let $\mathcal{G}$ be a Lie groupoid over M, and $\mathcal{O} \subset M$ an orbit through $x \in M$. If $\mathcal{G}$ is proper at x, then there are neighborhoods U and V of $\mathcal{O}$ such that $\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})|_V$.*

The first consequence is a stronger version of the result obtained by joining the works of Weinstein [18] and Zung [19]:

COROLLARY 1. – *If $\mathcal{G}$ is proper at x and $\mathcal{O}$ is of finite type, then one can find arbitrarily small neighborhoods U of $\mathcal{O}$ in M such that $\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})$.*

Of course, requiring U invariant is a natural condition. In this direction we have:

COROLLARY 2. – *If $\mathcal{G}$ is s-proper at x, then one can find arbitrarily small invariant neighborhood U of $\mathcal{O}$ in M such that $\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})$.*

Note that s-properness implies that $\mathcal{O}$ is compact. But also the property "one can find arbitrarily small invariant neighborhoods" alone is another strong property of $\mathcal{O}$ (called stability). Actually, if $\mathcal{G}$ is proper at x, then the stability of $\mathcal{O}$ forces s-properness at x. With the mind at proper actions by *non-compact* Lie groups, we address the following problem (where the orbit may be non-compact).

PROBLEM 0.1. – *If $\mathcal{G}$ is proper at x and s is trivial on an invariant neighborhood of x, under what extra-hypothesis on $\mathcal{O}$ (if any) does it follow that one can find an invariant neighborhood U of $\mathcal{O}$ in M such that $\mathcal{G}|_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})$.*

1. A more detailed introduction

1.1. Lie groupoid notations; properness

Throughout this paper, $\mathcal{G}$ will denote a Lie groupoid over a manifold M. Hence M is the manifold of objects, $\mathcal{G}$ is the manifold of arrows (in this paper all manifolds are assumed to be Hausdorff and second countable). We denote by $s, t : \mathcal{G} \longrightarrow M$ the source and the target maps and by gh the multiplication (composition) of arrows $g, h \in \mathcal{G}$ (defined when $s(g) = t(h)$).

DEFINITION 1.1. – Let $\mathcal{G}$ be a Lie groupoid over M, $x \in M$. We say that $\mathcal{G}$ is

- *proper* if the map $(s,t) : \mathcal{G} \longrightarrow M \times M$ is a proper map.
- *s-proper* if the map $s : \mathcal{G} \longrightarrow M$ is proper.
- *proper at x* if the map (s,t) is proper at (x,x), i.e., if any sequence $(g_n)_{n\geq 1}$ in $\mathcal{G}$ with $(s(g_n), t(g_n)) \to (x,x)$ admits a convergent subsequence.
- *s-proper at x* if the map s is proper at x.

REMARK 1.2. – It follows from Proposition 3.5 that properness at x implies properness at any point in the orbit $\mathcal{O}$ of x. Also, all the obvious implications above are strict. On the other hand, Ehresmann's theorem implies that, when the fibers of s are connected (which happens in many examples!), s-properness at x is equivalent to the compactness of $s^{-1}(x)$. A version of Ehresmann's theorem ("at x") implies that s-properness at x is actually equivalent to the condition that $s^{-1}(x)$ is compact and s is trivial around x.

EXAMPLE 1.3. – The main example to have in mind is the Lie groupoid associated to the action of a Lie group G on a manifold M. Known as the action Lie groupoid, and denoted $G \ltimes M$, it is a Lie groupoid over M whose manifold of arrows is $G \times M$, with source/target defined by $s(g,x) = x$, $t(g,x) = gx$ and the multiplication $(g,x)(h,y) = (gh,y)$ (defined when $x = hy$). The action groupoid $G \ltimes M$ is proper (or proper at x) if and only if the action of G on M is proper (or proper at $x \in M$, see e.g., [8]); s-properness corresponds to the compactness of G.

EXAMPLE 1.4. – For any submersion $\pi : X \longrightarrow Y$ one has a groupoid over X:

$$\mathcal{G}(\pi) = X \times_\pi X = \{(x,x') \in X \times X : \pi(x) = \pi(x')\},$$

with $s(x,y) = y$, $t(x,y) = x$ and multiplication $(x,y)(y,z) = (x,z)$. While $\mathcal{G}(\pi)$ is always proper, it is s-proper if and only if π is a proper map. When Y is a point, the resulting groupoid $X \times X$ is known as the pair groupoid over X (always proper, and s-proper if and only if X is compact).

EXAMPLE 1.5. – Associated to any principal G-bundle $\pi : P \longrightarrow M$ there is a Lie groupoid over M, known as the gauge groupoid of P, denoted Gauge(P), which is the quotient $(P \times P)/G$ of the pair groupoid of P modulo the diagonal action of G. It is proper if and only if G is compact; it is s-proper if and only if P is compact.

1.2. Orbits and the local model

Given a Lie groupoid $\mathcal{G}$ over M, two points $x, y \in M$ are in the same orbit of $\mathcal{G}$ if there is an arrow $g : x \longrightarrow y$ (i.e., $s(g) = x$, $t(g) = y$). This induces the partition of M by the *orbits* of $\mathcal{G}$. Each orbit carries a canonical smooth structure that makes it into an immersed submanifold of M (cf. e.g., [12], but see also below).

Let $\mathcal{O}$ be an orbit. The linearization theorem at $\mathcal{O}$ provides a "linear" model for $\mathcal{G}$ around $\mathcal{O}$. This model is just the tubular neighborhood in the world of groupoids, for $\mathcal{G}$ near $\mathcal{O}$. More precisely, over $\mathcal{O}$, $\mathcal{G}$ restricts to a Lie groupoid

$$\mathcal{G}_{\mathcal{O}} := \{g \in \mathcal{G} : s(g), t(g) \in \mathcal{O}\}.$$

Its normal bundle in $\mathcal{G}$ sits over the normal bundle of $\mathcal{O}$ in M:

$$\mathcal{N}_{\mathcal{O}}(\mathcal{G}) := T\mathcal{G}/T\mathcal{G}_{\mathcal{O}} \underset{dt}{\overset{ds}{\rightrightarrows}} \mathcal{N}_{\mathcal{O}} := TM/T\mathcal{O} \tag{1}$$

as a Lie groupoid. The groupoid structure is induced from the groupoid structure of the tangent groupoid $T\mathcal{G}$ (a groupoid over TM); i.e., the structure maps (source, the target and the multiplication) are induced by the differentials of those of $\mathcal{G}$.

There are various ways of realizing $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ more concretely. For instance, since $\mathcal{G}_{\mathcal{O}} = s^{-1}(\mathcal{O})$, its normal bundle in $\mathcal{G}$ is just the pull-back of $\mathcal{N}_{\mathcal{O}}$ by s:

$$\mathcal{N}_{\mathcal{O}}(\mathcal{G}) = \{(g, v) \in \mathcal{G}_{\mathcal{O}} \times \mathcal{N}_{\mathcal{O}} : s(g) = \pi(v)\}.$$

With this, the groupoid structure comes from a (fiberwise) linear action of $\mathcal{G}_{\mathcal{O}}$ on $\mathcal{N}_{\mathcal{O}}$. That means that any arrow $g : x \longrightarrow y$ in $\mathcal{G}_{\mathcal{O}}$ induces a linear isomorphism

$$g : \mathcal{N}_x \longrightarrow \mathcal{N}_y. \tag{2}$$

Explicitly: given $v \in \mathcal{N}_x$, one chooses a curve $g(t) : x(t) \longrightarrow y(t)$ in $\mathcal{G}$ with $g(0) = g$ and such that $\dot{x}(0) \in T_xM$ represents v, and then gv is represented by $\dot{y}(0) \in T_yM$. With these, the groupoid structure of $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ is given by

$$s(g, v) = v, \ \ t(g, v) = gv, \ \ (g, v)(h, w) = (gh, w).$$

Definition 1.6. – The Lie groupoid $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ is called *the linearization of $\mathcal{G}$ at $\mathcal{O}$*.

Example 1.7. – When x is a fixed point of $\mathcal{G}$, i.e., $\mathcal{O}_x = \{x\}$, then $\mathcal{N}_{\mathcal{O}_x}(\mathcal{G})$ is the groupoid associated (cf. Example 1.3) to the action (2) of G_x on $\mathcal{N}_x$.

The Lie groupoid $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$ can be further unravelled by choosing a point $x \in \mathcal{O}$. The outcome is a bundle-description of the linearization, which is closer to the familiar one from group actions. Here are the details. We fix $x \in \mathcal{O}$ and we use the notation $\mathcal{O} = \mathcal{O}_x$. Associated to x there are:

- the *s-fiber at x*, $P_x := s^{-1}(x)$, which is a submanifold of $\mathcal{G}$;
- the *isotropy group at x*, $G_x = s^{-1}(x) \cap t^{-1}(x)$ which is a Lie group with multiplication and smooth structure induced from the ones of $\mathcal{G}$;
- the *isotropy representation at x*, $\mathcal{N}_x = T_xM/T_x\mathcal{O}_x$, viewed as a representation of G_x with the action (2) described above;
- the *isotropy bundle at x*, which is P_x viewed as a principal G_x-bundle with the action induced by the multiplication of $\mathcal{G}$

$$t : P_x \longrightarrow \mathcal{O}_x. \tag{3}$$

It is this description that provides the orbit with its smooth structure: since the action of G_x on P_x is free and proper, it induces a smooth structure on $\mathcal{O}_x$, making (3) into a smooth bundle and $\mathcal{O}_x$ into an immersed submanifold of M. With these,

- the normal bundle $\mathcal{N}_{\mathcal{O}}$ is isomorphic to the associated vector bundle

  $$\mathcal{N}_{\mathcal{O}} \cong P_x \times_{G_x} \mathcal{N}_x,$$

  i.e., the quotient of $P_x \times \mathcal{N}_x$ modulo the action $\gamma \cdot (g, v) = (g\gamma^{-1}, \gamma v)$ of G_x;
- similarly, the space of arrows of the linearization is

  $$\mathcal{N}_{\mathcal{O}}(\mathcal{G}) \cong (P_x \times P_x) \times_{G_x} \mathcal{N}_x;$$

- In this new description of $\mathcal{N}_{\mathcal{O}}(\mathcal{G})$, the groupoid structure is given by

  $$s([p, q, v]) = [q, v], t([p, q, v]) = [p, v], [p, q, v] \cdot [q, r, v] = [p, r, v].$$