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FINITE DEGREES OF FREEDOM FOR THE REFINED BLOW-UP PROFILE OF THE SEMILINEAR HEAT EQUATION

BY VAN TIEN NGUYEN AND HATEM ZAAG

ABSTRACT. – We refine the asymptotic behavior of solutions to the semilinear heat equation with Sobolev subcritical power nonlinearity which blow up in some finite time at a blow-up point where the (supposed to be generic) profile holds. In order to obtain this refinement, we have to abandon the explicit profile function as a first order approximation, and take a non explicit function as a first order description of the singular behavior. This non explicit function is in fact a special solution which we construct, obeying some refined prescribed behavior. The construction relies on the reduction of the problem to a finite dimensional one and the use of a topological argument based on index theory to conclude. Surprisingly, the new non explicit profiles which we construct make a family with finite degrees of freedom, namely $\frac{N(N+1)}{2}$ if N is the dimension of the space.

RÉSUMÉ. – Nous raffinons le comportement asymptotique des solutions de l'équation semilinéaire de la chaleur avec une non-linéarité sous-critique au sens de Sobolev, qui explosent en temps fini à un point d'explosion avec le profil communément admis comme générique. Pour obtenir ce raffinement, nous devons abandonner le profil explicite comme premier ordre de l'approximation, et prenons à la place une fonction non explicite comme première description du comportement au voisinage de la singularité. Cette fonction non explicite est en fait une solution spécifique que nous construisons, obéissant à un certain comportement prescrit. La construction repose sur la réduction du problème à un problème en dimension finie et l'utilisation d'un argument topologique basé sur la théorie du degré pour conclure. De façon étonnante, on constate que le nouveau profil non explicite produit une famille avec un nombre fini de degrés de liberté, soit $\frac{(N+1)N}{2}$ si N est la dimension de l'espace.

1. Introduction

We are interested in the following semilinear heat equation:

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \in L^\infty(\mathbb{R}^N), \end{cases} \tag{1}$$

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where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, Δ denotes the Laplacian in $\mathbb{R}^N$, and

$$p > 1 \quad \text{or} \quad 1 < p < \frac{N+2}{N-2} \quad \text{if} \quad N \geq 3.$$

Equation (1) is a simple model for a large class of nonlinear parabolic equations. In fact, it captures features common to a whole range of blow-up problems parsing in various physical situations, particularly it highlights the role of scaling and self-similarity. Among related equations, we would like nonetheless to mention: the solid fuel ignition model (Bebernes, Bressan and Eberly [2]), the thermal explosion (Bebernes and Kassoy [3], Kassoy and Poland [24]), [25]), surface diffusion (Bernoff, Bertozzi and Witelski [4]), the motion by mean curvature (Soner and Souganidis [40]), vortex dynamics in superconductors (Chapman, Hunton and Ockendon [8], Merle and Zaag [29]).

By standard results, the problem (1) has a unique classical solution $u(x,t)$ continuous in time with values in $L^\infty(\mathbb{R}^N)$, which exists at least for small times. The solution $u(x,t)$ may develop singularities in some finite time (see Kaplan [23], Fujita [15], Levine [26], Ball [1], Weissler [45] for the existence of finite-time blow-up solutions to (1)). In this case, we say that $u(x,t)$ blows up in a finite time $T < +\infty$ in the sense that

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

Here we call T the blow-up time of $u(x,t)$. In such a blow-up case, we say that $\hat{a} \in \mathbb{R}^N$ is a blow-up point of u if u is not locally bounded in the neighborhood of $(\hat{a}, T)$, this means that there exists $(x_n, t_n) \to (\hat{a}, T)$ such that $|u(x_n, t_n)| \to +\infty$ when $n \to +\infty$.

Let us consider $u(t)$ a solution of (1) which blows up in finite time T at only one blow-up point $\hat{a}$. From the translation invariance of (1), we may assume that $\hat{a} = 0$. Studying the solution $u(x,t)$ near the singularity $(0,T)$ is based on the following *similarity variables* (see [17, 18]):

$$\mathscr{T}[u](y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T-t), \tag{2}$$

and $w = \mathscr{T}[u]$ solves a new parabolic equation in (y,s),

$$\partial_s w = \mathscr{L} w - \frac{p}{p-1} w + |w|^{p-1} w, \quad (y,s) \in \mathbb{R}^N \times [-\log T, +\infty), \tag{3}$$

where

$$\mathscr{L} = \Delta - \frac{y}{2} \cdot \nabla + 1. \tag{4}$$

In view of (2), the study of $u(x,t)$ as $(x,t) \to (0,T)$ is then equivalent to the study of $\mathscr{T}[u](y,s)$ as $s \to +\infty$, and each result for u has an equivalent formulation in term of $\mathscr{T}[u]$.

According to Giga and Kohn in [18] (see also [16, 17]), we know that:

If $\hat{a}$ is a blow-up point of u, then

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(\hat{a} + y\sqrt{T-t}, t) = \lim_{s \to +\infty} \mathscr{T}[u](y,s) = \pm\kappa, \tag{5}$$

uniformly on compact sets $|y| \leq R$, where $\kappa = (p-1)^{-\frac{1}{p-1}}$.

The estimate (5) has been refined until the higher order by Filippas, Kohn and Liu [13], [14], Herrero and Velázquez [20], [22], [41], [43], [42]. More precisely, they classified the behavior of $\mathscr{T}[u](y,s)$ for $|y|$ bounded, and showed that one of the following cases occurs (up to replacing u by $-u$ if necessary),

- *Case 1 (non-degenerate rate of blow-up): There exists $\ell \in \{1,\dots,N\}$, and up to an orthogonal transformation of space coordinates,*

$$\forall R > 0,\ \sup_{|y|\le R}\left|\mathscr{T}[u](y,s) - \left[\kappa + \frac{\kappa}{4ps}\left(2\ell - \sum_{i=1}^{\ell}|y_i|^2\right)\right]\right| = \mathcal{O}\left(\frac{\log s}{s^2}\right). \tag{6}$$

- *Case 2 (degenerate rate of blow-up): There exists $\mu > 0$ such that*

$$\forall R > 0,\ \sup_{|y|\le R}|\mathscr{T}[u](y,s) - \kappa| = \mathcal{O}(e^{-\mu s}), \tag{7}$$

(this exponential convergence has been refined up to the order 1 by Herrero and Velázquez, but we omit that description since we choose in this work to concentrate on the non-degenerate rate of blow-up mentioned in the case 1 above).

If $\ell = N$, then $\hat{a} = 0$ is an isolated blow-up point from Velázquez [41]. Merle and Zaag [31, 32, 33] (with no sign condition), and Herrero and Velázquez [41, 22] (in the positive case) established the following blow-up profile in the variable $\xi = \frac{y}{\sqrt{s}}$ (which is the intermediate scale that separates the regular and singular parts in the non-degenerate case):

$$\forall R > 0,\quad \sup_{|\xi|\le R}\left|\mathscr{T}[u](\xi\sqrt{s},s) - f(\xi)\right| \to 0 \quad \text{as } s \to +\infty, \tag{8}$$

where

$$f(\xi) = \kappa\left(1 + \frac{p-1}{4p}|\xi|^2\right)^{-\frac{1}{p-1}}. \tag{9}$$

Herrero and Velázquez [21] proved that the profile (9) is generic in the case $N = 1$, and they announced the same for $N \ge 2$, but they never published it.

Merle and Zaag [31], [32], [33] derived the limiting profile in the $u(x,t)$ variable, in sense that $u(x,t) \to u^*(x)$ when $t \to T$ if $x \neq 0$ and x is the neighborhood of 0, with

$$u^*(x) \sim \left[\frac{8p|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}} \qquad \text{as } x \to 0. \tag{10}$$

They also showed that all the behaviors (6) with $\ell = N$, (8) and (10) are equivalent.

Bricmont and Kupiainen [7], Merle and Zaag in [30] showed the existence of initial data for (1) such that the corresponding solutions blow up in finite time T at only one blow-up point $\hat{a} = 0$ and verify the behavior (8). Note that the method of [30] allows to derive the stability of the blow-up behavior (8) with respect to perturbations in the initial data or the nonlinearity (see also Fermanian, Merle and Zaag [11], [12] for other proofs of the stability).

In this work, considering the expansions (6) with $\ell = N$, (8) and (10), we ask whether we can carry on these expansions and obtain lower order estimates. In particular in (10), we ask whether we can obtain the following terms of the expansion, up to bounded functions? In view of the self-similar transformation (2), a necessary condition would be to carry on the expansion (6) up to the scale of $e^{-\frac{s}{p-1}} = (T-t)^{\frac{1}{p-1}}$. Unfortunately, any attempt to carry on the expansion (6) would give bunches of terms in the scale of powers of $\frac{1}{s} = \frac{1}{|\log(T-t)|}$ (with