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CURVE TEST FOR ENHANCED IND-SHEAVES AND HOLONOMIC D-MODULES, I

BY TAKURO MOCHIZUKI

Dedicated to Professor Masaki Kashiwara on the occasion of his 75th birthday

ABSTRACT. – Recently, the Riemann-Hilbert correspondence was generalized to the context of holonomic D-modules by A. D'Agnolo and M. Kashiwara. Namely, they proved that their enhanced de Rham functor induces a fully faithful embedding of the derived category of cohomologically holonomic complexes of D-modules into the derived category of complexes of cohomologically $\mathbb{R}$-constructible enhanced ind-sheaves.

In this series of papers, we study a condition when a complex of $\mathbb{R}$-constructible enhanced ind-sheaves K is induced by a cohomologically holonomic complex of D-modules. It is our goal to characterize such K in terms of the restriction of K to holomorphic curves. In this paper (part I), as a preliminary, we shall study some issues for multi-sets of subanalytic functions.

RÉSUMÉ. – Récemment, la correspondance de Riemann-Hilbert a été généralisée au contexte des D-modules holonomes par A. D'Agnolo et M. Kashiwara. En effet, ils ont prouvé que leur foncteur de de Rham renforcé induit un plongement pleinement fidèle de la catégorie dérivée des complexes de D-modules à cohomologie holonome dans la catégorie dérivée des complexes d'ind-faisceaux renforcés à cohomologie $\mathbb{R}$-constructible.

Dans cette série d'articles, nous étudions une condition lorsqu'un complexe d'ind-faisceaux renforcé à cohomologie $\mathbb{R}$-constructibles K est induit par un complexe des D-modules à cohomologie holonomes. Notre objectif est de caractériser un tel K en termes de restriction de K aux courbes holomorphes. Dans ce document (partie I), en guise de préparation, nous étudierons certains problèmes pour les multi-ensembles de fonctions sous-analytiques.

1. Introduction

1.1. Goal of the study

In [2], D'Agnolo and Kashiwara established the Riemann-Hilbert correspondence for holonomic $\mathscr{D}$-modules, by generalizing the classical Riemann-Hilbert correspondence

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[10, 23, 22] between complexes of regular holonomic $\mathscr{D}$-modules and cohomologically $\mathbb{C}$-constructible complexes. They introduced the concept of $\mathbb{R}$-constructible enhanced ind-sheaves on the basis of the theory of ind-sheaves [13, 14]. For any complex manifold X, they constructed the enhanced de Rham functor $\mathrm{DR}_X^{\mathsf{E}}$ from the derived category of cohomologically holonomic complexes of $\mathscr{D}_X$-modules $\mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_X)$ to the derived category of $\mathbb{R}$-constructible enhanced ind-sheaves $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, and they proved that $\mathrm{DR}_X^{\mathsf{E}}$ is fully faithful and compatible with the 6-operations and the duality. They also gave the reconstruction of a cohomologically holonomic complex of $\mathscr{D}_X$-modules $\mathscr{M}^\bullet$ from its solution complex $\mathrm{Sol}_X^{\mathsf{E}}(\mathscr{M}^\bullet) = \mathrm{DR}_X^{\mathsf{E}}(\boldsymbol{D}_X\mathscr{M}) \in \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, where $\boldsymbol{D}_X$ denotes the duality functor on $\mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_X)$. They efficiently used the study of the formal structure and the asymptotic analysis of meromorphic flat bundles ([16], [17], [20], [26], [27], [32]). In [3], they also introduced the natural perversity condition for $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, and they proved that $\mathrm{DR}_X^{\mathsf{E}}$ is exact with respect to the natural t-structure of $\mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_X)$ and the t-structure of $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ with respect to the perversity condition. (See [6, 11, 15] for more details.)

We may still ask an interesting question. Let $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$ denote the essential image of $\mathrm{DR}_X^{\mathsf{E}}$. It is natural to ask a condition for an object $K \in \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ to be an object in $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$. In the regular singular case, it is the cohomological $\mathbb{C}$-constructibility condition defined in a microlocal way. As far as the author knows, such a clear condition has not yet been given in the enhanced case.

In this paper, we study "a curve test". We consider the full subcategory $\mathsf{E}^b_{\triangle}(\mathrm{I}\mathbb{C}_X)$ of $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, determined by the following condition for objects $K \in \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$.

— Set $\Delta := \{|z| < 1\}$. Let $\varphi : \Delta \longrightarrow X$ be any holomorphic map. Then we have $\mathsf{E}\varphi^{-1}(K) \in \mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_\Delta)$.

By the compatibility of the enhanced de Rham functor DR^{E} and 6-operations, $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$ is clearly a full subcategory of $\mathsf{E}^b_{\triangle}(\mathrm{I}\mathbb{C}_X)$. In this paper and [30], we shall prove the following theorem.

THEOREM 1.1. – *$\mathsf{E}^b_{\triangle}(\mathrm{I}\mathbb{C}_X)$ is equal to $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$.*

REMARK 1.2. – Kuwagaki [19] introduced another formulation for the irregular Riemann-Hilbert correspondence. The essential image $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$ is also studied by Ito in [9].

1.1.1. *Meromorphic flat connections and enhanced ind-sheaves.* – A holonomic $\mathscr{D}$-module can be locally described as the gluing of meromorphic flat connections on complex analytic subspaces. Hence, it is a key step to study such a characterization for meromorphic flat connections.

Let X be an n-dimensional complex manifold. Let H be a normal crossing hypersurface of X. Let $\boldsymbol{X}(H)$ denote the bordered space $(X \setminus H, X)$ in the sense of [2, 3]. (A bordered space is defined to be a pair $(M, \check{M})$ of a good topological space $\check{M}$ and an open subset M, where a topological space is called good if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension. See [2, Page 86].) We obtain the derived category $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ of $\mathbb{R}$-constructible enhanced ind-sheaves on $\boldsymbol{X}(H)$ as the quotient of $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ by $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_H)$. We obtain the full subcategory $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)}) \subset \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$

of the objects obtained as $\mathrm{DR}^{\mathsf{E}}_{\boldsymbol{X}(H)}(V)[-n]$ for meromorphic flat connections (V, ∇) on (X, H).

We set $\Delta^* := \Delta \setminus \{0\}$. Let $\boldsymbol{X}(H)^{\boldsymbol{\Delta}(0)}$ be the set of holomorphic maps $\varphi : \Delta \longrightarrow X$ satisfying $\varphi(\Delta^*) \subset X \setminus H$ and $\varphi(0) \in H$.

We obtain the full subcategory $\mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)}) \subset \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ of the objects K satisfying the following conditions.

— $K|_{X\setminus H}$ is induced by a local system on $X \setminus H$.

— For any $\varphi \in \boldsymbol{X}(H)^{\boldsymbol{\Delta}(0)}$, $\mathsf{E}\varphi^{-1}(V)$ is an object in $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{\Delta}(0)})$.

Clearly, $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ is an object in $\mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$. The following is the essential part in the proof of Theorem 1.1.

THEOREM 1.3. – *We have* $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)}) = \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$.

In other words, for any $K \in \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$, we would like to prove that there exists a meromorphic flat connection (V, ∇) on (X, H) with an isomorphism $\mathrm{DR}^{\mathsf{E}}_{\boldsymbol{X}(H)}(V)[-n] \simeq K$ in $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$.

1.2. Structure of meromorphic flat connections

To explain the ideas of the proof of Theorem 1.3, let us review the structure of meromorphic flat connections. See surveys (for example, [26] and [33]) for a more detailed explanation.

Recall that, in general, a set I with a map $\mathfrak{m} : I \longrightarrow \mathbb{Z}_{>0}$ is called a multi-set. A multi-subset of a set S is defined to be a subset $I \subset S$ with a map $I \longrightarrow \mathbb{Z}_{>0}$. If I is finite, the multi-subset $(I, \mathfrak{m})$ is called finite.

1.2.1. *One dimensional case*. – Let $\mathcal{O}_\Delta$ be the sheaf of holomorphic functions on Δ, and let $\mathcal{O}_\Delta(*0)$ denote the sheaf of meromorphic functions with poles along 0. For any sheaf $\mathcal{F}$ on Δ, let $\mathcal{F}_0$ denote the stalk of $\mathcal{F}$ at 0. For any positive integer e, we set $\mathcal{O}^{(e)}_{\Delta,0} = \mathcal{O}_{\Delta,0}[z^{1/e}]$ and $\mathcal{O}_\Delta(*0)^{(e)}_0 = \mathcal{O}_\Delta(*0)_0[z^{1/e}]$, where $z^{1/e}$ denotes an e-th root of the variable z. Let $G^{(e)}$ be the Galois group of the extensions. Moreover, let $\mathcal{O}^{(e)}_{\Delta,\widehat{0}}$ denote the completion of the local ring $\mathcal{O}^{(e)}_{\Delta,0}$, i.e., the ring of formal power series $\sum_{j=0}^{\infty} a_j z^{j/e}$ $(a_j \in \mathbb{C})$.

Let (V, ∇) be a meromorphic flat bundle on $(\Delta, 0)$. According to the Hukuhara-Levelt-Turrittin theorem, there exist a positive integer e, a unique $G^{(e)}$-invariant subset $\mathrm{Irr}(V, \nabla) \subset \mathcal{O}_\Delta(*0)^{(e)}_0 / \mathcal{O}^{(e)}_{\Delta,0}$ and a decomposition

$$(V, \nabla) \otimes_{\mathcal{O}_\Delta} \mathcal{O}^{(e)}_{\Delta,\widehat{0}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(V,\nabla)} (\widehat{V}_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$$

such that (i) $V_{\mathfrak{a}} \neq 0$, (ii) $\nabla_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}\,\mathrm{id}_{\widehat{V}_{\mathfrak{a}}}$ are regular singular, where $\widetilde{\mathfrak{a}} \in \mathcal{O}_X(*H)^{(e)}_0$ are representatives of $\mathfrak{a}$. Thus, we obtain a multi-subset $(\mathrm{Irr}(V, \nabla), \mathrm{rank})$ of $\mathcal{O}_\Delta(*0)^{(e)}_0 / \mathcal{O}^{(e)}_{\Delta,0}$, called the multi-set of the irregular values of (V, ∇). Let $\widetilde{\mathrm{Irr}}(V, \nabla) \subset \mathcal{O}_\Delta(*0)^{(e)}_0$ be a lift of $\mathrm{Irr}(V, \nabla)$, i.e., the projection $\mathcal{O}_\Delta(*0)^{(e)}_0 \longrightarrow \mathcal{O}_\Delta(*0)^{(e)}_0 \big/ \mathcal{O}^{(e)}_{\Delta,0}$ induces a bijection $\widetilde{\mathrm{Irr}}(V, \nabla) \longrightarrow \mathrm{Irr}(V, \nabla)$. The pre-image of $\mathfrak{a}$ in $\widetilde{\mathrm{Irr}}(V, \nabla)$ is denoted by $\widetilde{\mathfrak{a}}$. By setting $\mathrm{rank}(\widetilde{\mathfrak{a}}) := \mathrm{rank}(\mathfrak{a})$, we obtain the multi-set $(\widetilde{\mathrm{Irr}}(V, \nabla), \mathrm{rank})$.