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GENUS TWO CURVES ON ABELIAN SURFACES

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ABSTRACT. – This paper deals with singularities of genus 2 curves on a general (d_1, d_2)-polarized abelian surface (S, L). In analogy with Chen's results concerning rational curves on K3 surfaces [6, 7], it is natural to ask whether all such curves are nodal. We prove that this holds true if and only if d_2 is not divisible by 4. In the cases where d_2 is a multiple of 4, we exhibit genus 2 curves in $|L|$ that have a triple, 4-tuple or 6-tuple point. We show that these are the only possible types of unnodal singularities of a genus 2 curve in $|L|$. Furthermore, with no assumption on d_1 and d_2, we prove the existence of at least one nodal genus 2 curve in $|L|$. As a corollary, we obtain nonemptiness of all Severi varieties on general abelian surfaces and hence generalize [18, Thm. 1.1] to nonprimitive polarizations.

RÉSUMÉ. – Cet article traite des singularités des courbes de genre 2 sur une surface abélienne polarisée de type (d_1, d_2) générale. Par analogie avec les résultats de Chen concernant les courbes rationelles sur les surfaces $K3$ [6, 7], il est naturel de se demander si toutes ces courbes sont nodales. Nous démontrons que c'est bien le cas si et seulement si d_2 n'est pas divisible par 4. Dans le cas où d_2 est un multiple de 4, nous exhibons des courbes de genre 2 dans $|L|$ ayant un point triple, quadruple ou sextuple. Nous démontrons que ce sont les seules singularités non nodales possibles pour une courbe de genre 2 dans $|L|$. En outre, sans hypothèse sur d_1 et d_2, nous démontrons l'existence d'au moins une courbe nodale de genre 2 dans $|L|$. On obtient en corollaire que toutes les varietés de Severi sur une surface abélienne générale sont non vides, généralisant ainsi [18, Thm. 1.1] aux polarisations non-primitives.

1. Introduction

The minimal geometric genus of any curve lying on a general abelian surface is 2 and there are finitely many curves of such genus in a fixed linear system. The role of genus two curves on abelian surfaces is thus analogous to that of rational curves on $K3$ surfaces, but until now it has not been investigated as extensively. Their enumeration is now well understood. Their count in the primitive case was carried out by Göttsche [15], Debarre [8] and Lange-Sernesi [20], and used in [8] in order to compute the Euler characteristic of generalized Kummer varieties. Only recently, Bryan, Oberdieck, Pandharipande and Yin [4] handled the

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nonprimitive case, thus obtaining a formula parallel to the full Yau-Zaslow conjecture for rational curves on K3 surfaces (cf. [17]).

Singularities of rational curves on $K3$ surfaces have received plenty of attention. Mumford [24, Appendix] first proved the existence of a nodal rational curve in the primitive linear system $|L|$ on a general genus g polarized $K3$ surface (S, L); as a byproduct, he obtained nonemptiness of the Severi variety $|L|_\delta$ parametrizing δ-nodal curves in $|L|$ for any integer $0 \leq \delta \leq g$. His results were then generalized by Chen [6, 7] to nonprimitive linear systems. In the primitive case, Chen managed to deal with all rational curves in $|L|$ showing that they are all nodal; the analogue for nonprimitive linear systems is still an open problem.

Singularities of genus 2 curves on abelian surfaces are not as well understood, even though they are necessarily ordinary (cf. [21, Prop. 2.2]). The natural question whether any genus 2 curve on a general (d_1, d_2)-polarized abelian surface is nodal [22, Pb. 2.7] has negative answer if one does not make any assumption on d_1 and d_2. Indeed, multiplication by 2 on a principally polarized abelian surface (A, L) identifies the six Weierstrass points of its theta divisor, whose image is thus a genus 2 curve with a 6-tuple point lying in (a translate of) the linear system $|L^{\otimes 4}|$ (cf. Example 2). Since this is a polarization of type $(4, 4)$, all genus 2 curves may still be expected to be nodal in primitive linear systems (or even in linear systems not divisible by 4, cf. [22, Conj. 2.10]). Our main result is that this expectation does not hold in its full generality and detects all the cases where it fails, thus completely answering the question.

THEOREM 1.1. – *Let (S, L) be a general abelian surface with a polarization of type (d_1, d_2). Then any genus 2 curve in the linear system $|L|$ is nodal if and only if 4 does not divide d_2.*

When d_2 is a multiple of 4, we exhibit genus 2 curves in $|L|$ that have an unnodal singularity and, more precisely, a triple, a 4-tuple or a 6-tuple point (cf. Examples 1 and 2). We also show that these are the only types of unnodal singularities that a genus 2 curve on a general abelian surface may acquire (cf. Remark 1). To our knowledge, the best bound on the order of such a singularity in the literature was $\frac{1}{2}\left(1 + \sqrt{8d_1d_2 - 7}\right)$ by Lange-Sernesi, cf. [21, Prop. 2.2]. The existence of unnodal genus 2 curves in all primitive linear systems of type $(1, 4k)$ is quite striking and highlights a major difference with the $K3$ case.

When 4 divides d_2, the above theorem does not exclude the existence in $|L|$ of some nodal genus 2 curves. This is indeed proved in the following:

THEOREM 1.2. – *Let (S, L) be a general (d_1, d_2)-polarized abelian surface. Then the linear system $|L|$ contains a nodal curve of genus 2.*

Given a nodal genus 2 curve as above, standard deformation theory enables one to smooth any of its nodes independently remaining inside the linear system $|L|$. As a consequence, Theorem 1.2 yields nonemptiness of all Severi varieties on general abelian surfaces:

COROLLARY 1.3. – *Let (S, L) be a general (d_1, d_2)-polarized abelian surface. Then, for any $0 \leq \delta \leq d_1d_2 - 1$ the Severi variety $|L|_\delta$ is nonempty and smooth of dimension equal to $d_1d_2 - 1 - \delta$.*

This generalizes [18, Thm. 1.1] to nonprimitive linear systems. Note that, since S has trivial canonical bundle, the regularity of $|L|_\delta$ stated in Corollary 1.3 follows for free from its nonemptiness by the proofs of Propositions 1.1 and 1.2 in [21]. We mention that the irreducible components of the Severi varieties on a general primitively polarized abelian surface have been determined very recently by Zahariuc in [29].

We now spend some words on the proofs of Theorems 1.1 and 1.2. In contrast to the methods proposed in [6, 7, 18], we need neither to degenerate S to a singular surface nor to specialize it to an abelian surface with large Neron-Severi group. Instead, we exploit the universal property of Jacobians in order to translate the if part of Theorem 1.1 and Theorem 1.2 into the following statement concerning Brill-Noether theory on a general curve of genus 2:

THEOREM 1.4. – *Let $[C] \in \mathcal{M}_2$ be a general genus 2 curve and fix any integer $d \geq 4$. If C admits a g^2_d totally ramified at three points P_1, P_2, P_3, then d is even and P_1, P_2, P_3 are Weierstrass points.*

We refer to Section §2 for the details of this reduction, that we mention here only briefly. The key fact is that any genus 2 curve $\overline{C}$ on a (d_1, d_2)-polarized abelian surface S with normalization C arises as image of a composition

$$C \overset{u}{\hookrightarrow} J(C) \overset{\lambda}{\to} S, \tag{1}$$

where u is the Abel-Jacobi map and λ is an isogeny. Three points $P_1, P_2, P_3 \in C = u(C)$ identified by λ necessarily differ by elements in its kernel. Since the order of any such element is divisible by d_2, the three divisors $d_2P_1, d_2P_2, d_2P_3 \in C_d$ are linearly equivalent and thus define (for $d_2 \geq 4$) a $g^2_{d_2}$ on C totally ramified at three points. Theorem 1.4 excludes the existence of such a linear series for C general and odd values of d_2, thus implying our main results in these cases. If instead d_2 is even, a $g^2_{d_2}$ totally ramified at three points does exist: as soon as P_1, P_2, P_3 are Weierstrass points of C, the divisors $2P_1, 2P_2, 2P_3$ are linearly equivalent and thus the same holds true for d_2P_1, d_2P_2, d_2P_3. Conversely, by Theorem 1.4, any $g^2_{d_2}$ with three points of total ramification on C is of this type. This characterization is used in Section §2 both to prove the if part of Theorem 1.1 and Theorem 1.2 for $d_2 \equiv 2 \bmod 4$, and to provide examples of genus 2 curves with a triple, 4-tuple or 6-tuple point (cf. Examples 1 and 2) when $d_2 \equiv 0 \bmod 4$ implying the only if part of Theorem 1.1. These examples are based on the construction of suitable isogenies λ as in (1) or, equivalently by taking their kernels, suitable isotropic (with respect to the commutator pairing) subgroups of the group $J(C)[d_1d_2]$ of d_1d_2-torsion points of $J(C)$.

Section §3 is devoted to the proof of Theorem 1.4. This is done in two steps. First, we degenerate C to the transversal union of two elliptic curves meeting at a point and reduce Theorem 1.4 into a statement of Brill-Noether theory with ramification on a general elliptic curve (cf. Proposition 3.1). This reduction seriously involves the theory of limit linear series on curves of compact type, for which we refer to the original papers by Eisenbud and Harris [11, 12, 13]. Proposition 3.1 is then proved by an infinitesimal study of a *generalized Severi variety* (cf. [5, 28]).