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COCYCLES OVER PARTIALLY HYPERBOLIC MAPS

by

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1. Partially hyperbolic diffeomorphisms

A diffeomorphism $f\colon M \to M$ on a compact manifold M is *partially hyperbolic* if there exists a continuous, nontrivial Df-invariant splitting

$$T_xM = E^s_x \oplus E^c_x \oplus E^u_x, \quad x \in M$$

of the tangent bundle such that the derivative is a contraction along E^s and an expansion along E^u, with uniform rates, and the behavior of Df along the *center bundle* E^c is in between its behaviors along E^s and E^u, again by a uniform factor. Partial hyperbolicity is a natural generalization of the notion of uniform hyperbolicity (Anosov or even Axiom A, see [**25**]), that includes many interesting additional examples, most notably: diffeomorphisms derived from Anosov through deformation by isotopy, many affine maps on homogeneous spaces, certain skew-products over hyperbolic maps, and time-1 maps of Anosov flows. Partial hyperbolicity is an open condition, so any C^1 small perturbation of these examples is partially hyperbolic as well.

The *stable* and *unstable bundles*, E^s and E^u, are uniquely integrable; that is, there exist unique f-invariant foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ tangent to E^s and E^u, respectively, at all points. The leaves of these foliations are C^k if the diffeomorphism is C^k, for any $1 \le k \le \infty$, but the foliations are usually not transversely smooth. On the other hand, if f is twice differentiable then each $\mathcal{W}^s$ and $\mathcal{W}^u$ is absolutely continuous, meaning that its holonomy maps preserve the class of zero Lebesgue measure sets. These facts go back to the pioneering work of Brin, Pesin [**6**] and Hirsch, Pugh, Shub [**15**] where partial hyperbolicity and the closely related notion of normally hyperbolic foliations were introduced.

In general, the center bundle E^c need not be integrable, and similarly for the center stable bundle $E^{cs} = E^c \oplus E^s$ and the center unstable bundle $E^{cu} = E^c \oplus E^u$. We call the diffeomorphism *dynamically coherent* if E^{cs} and E^{cu} are tangent to foliations

$\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ respectively. Then intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, one obtains an integral foliation $\mathcal{W}^{c}$ for the center bundle as well. As it turns out, dynamical coherence does hold in many situations of interest.

Brin, Pesin **[6]** also introduced the notion of accessibility, which has played a central role in recent developments. A partially hyperbolic diffeomorphism is called *accessible* if any two points in the ambient manifold may be joined by an *su-path*, that is, a piecewise smooth path such that every smooth subpath is contained in a single leaf of $\mathcal{W}^{s}$ or a single leaf of $\mathcal{W}^{u}$. More generally, the diffeomorphism is *essentially accessible* if, given any two sets with positive volume, one can join some point of one to some point of the other by an *su*-path.

Interest in partially hyperbolic systems was greatly renewed in the mid-nineties, with two initial goals in mind. One goal was to characterize robust (or stable) transitivity, both in discrete time and continuous time. A dynamical system is *transitive* if it possesses orbits that are dense in the whole ambient space. The best known examples are all of the known constructions of Anosov diffeomorphisms (see **[25]**). Actually, since Anosov maps form an open subset of all C^1 diffeomorphisms, these are also examples of *robust* transitivity. On the other hand, early constructions by Shub **[24]** and Mañé **[17]** showed that diffeomorphisms can be robustly transitive without being Anosov. Many other examples were found by Bonatti, Díaz **[2]** and Bonatti, Viana **[5]**. A subsequent series of works started by Díaz, Pujals, Ures **[10]** for diffeomorphisms, and Morales, Pacifico, Pujals **[18]** for flows, established that in dimension three robustness implies partial hyperbolicity (where at least two of the bundles in the partially hyperbolic splitting are non-trivial). In higher dimensions one has to replace partial hyperbolicity by a related weaker condition called existence of a dominated splitting. See **[3, 5]** and also [**4**, Chapter 7] and references therein.

Another goal, initiated by Grayson, Pugh, Shub **[14]**, was to recover the original attempt by Brin, Pesin **[6]** to prove that most partially hyperbolic, volume preserving diffeomorphisms are actually ergodic. To this end, Pugh, Shub **[20]** proposed the following pair of conjectures:

Conjecture 1. — Accessibility holds for an open and dense subset of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.

Conjecture 2. — A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.

Concerning Conjecture 1, it was shown by Dolgopyat, Wilkinson **[12]** that accessibility holds for a C^1-open and dense subset of all partially hyperbolic diffeomorphisms, volume preserving or not. Moreover, Didier **[11]** proved that accessibility is C^1-open for systems with 1-dimensional center bundle. More recently, Rodriguez Hertz, Rodriguez Hertz, Ures **[23]** verified the complete conjecture for conservative systems whose center bundle is one-dimensional: accessibility is C^r-dense among C^r partially hyperbolic diffeomorphisms, for any $r \geq 1$. A version of this statement for non-conservative diffeomorphisms was obtained in [**7**]. It remains open whether C^r-density still holds when $\dim E^c > 1$.

Partial versions of Conjecture 2 were obtained by Pugh, Shub [**20, 21, 22**], assuming dynamical coherence and an additional technical condition they called center bunching. Roughly speaking, their notion of center bunching means that the diffeomorphism is close to being an *isometry* along center leaves. The best result to date on Conjecture 2 is due to Burns, Wilkinson [**8**] who proved ergodicity for any accessible, partially hyperbolic volume preserving diffeomorphism (not necessarily dynamically coherent) which is not too far from being *conformal* along center leaves. Although this property is also called center bunching, it is a lot milder than the one of Pugh, Shub. In particular, it is automatic when E^c has dimension one. Thus, the previous result contains as a corollary a complete proof of Conjecture 2 when the center bundle is one-dimensional. This corollary was also observed in [**23**].

2. Cocycles

The problems considered in this volume are situated in the following context. Let $f\colon M \to M$ be a diffeomorphism. We fix a (topological, Lie...) group H with identity element e and consider the set of all (continuous, Hölder continuous, smooth...) functions $\phi\colon M \to H$. Such a function is called a *cocycle*, for reasons that are explained in the sequel. Cocycles are objects that can be composed along orbits of f, and indeed, by the cocycle generated by ϕ we often mean the sequence ϕ_n defined by

$$\phi_n(x) = \begin{cases} \phi(f^{n-1}(x)) \cdots \phi(f(x)) \cdot \phi(x) & \text{if } n > 0, \\ \phi^{-1}(f^{-n}(x)) \cdots \phi^{-1}(f^{-2}(x)) \cdot \phi^{-1}(f^{-1}(x)) & \text{if } n < 0, \\ e & \text{if } n = 0. \end{cases}$$

An equivalent definition of a cocycle, and one that generalizes to actions of groups other than $\mathbb{Z}$, is the following. A 1-cocycle is a map $\alpha\colon \mathbb{Z} \times M \to H$ satisfying the *cocycle condition:*

$$\alpha(m+n, x) = \alpha(m, f^n(x)) \cdot \alpha(n, x), \qquad \forall n, m \in \mathbb{Z},\ x \in M. \tag{1}$$

Setting $\phi(x) = \alpha(1, x)$, we obtain from the cocycle condition that $\phi_n(x) = \alpha(n, x)$, thereby establishing the equivalence of the two notions.

There are several contexts in which cocycles arise immediately in smooth dynamics and related topics, which we now discuss.

Abelian cocycles. — The cocycle ϕ is called *abelian* when the group H is abelian. A fundamental example of an abelian cocycle is the Jacobian map $\operatorname{Jac} f\colon M \to \mathbb{R}_*$ that measures the volume distortion of a diffeomorphism $f\colon M \to M$ on a Riemannian manifold M:

$$\operatorname{Jac} f(x) = \frac{d(\mathrm{vol} \circ f)}{d\,\mathrm{vol}}(x).$$

The 1-cocycle generated by $\operatorname{Jac} f$ is $\alpha(n,x) = \operatorname{Jac} f^n(x)$; in this case the cocycle condition amounts to the composition law for Radon-Nikodym derivatives. Usually this cocycle is transformed to an additive cocycle by taking a logarithm: $\log \operatorname{Jac} f \colon M \to \mathbb{R}$.

Abelian cocycles appear more generally as potentials in thermodynamic formalism. In this setup, one associates to each cocycle $\phi\colon M \to \mathbb{R}$ over a dynamical system $f\colon M \to M$ one or more f-invariant probability measures μ_ϕ satisfying the variational equation

$$\int_M \phi \, d\mu_\phi + h(\mu_\phi) = \sup_\nu \left(\int_M \phi \, d\nu + h(\nu) \right),$$

where the supremum on the right is taken over all f-invariant probability measures ν, and $h(\nu)$ denotes the f-entropy of the measure ν. The functional

$$P(\phi) = \sup_\nu \left(\int_M \phi \, d\nu + h(\nu) \right),$$

called the *pressure* of ϕ, has the property that if

$$\phi - \psi = \Phi \circ f - \Phi, \tag{2}$$

for some function Φ, then $P(\phi) = P(\psi)$. Hence the measure μ_ϕ depends only on the equivalence equivalence class for the equivalence relation $\phi \sim \psi$ if and only if (2) holds. As we describe below, this equation can be viewed as a coboundary equation in the appropriate cohomology theory.

Another place in which abelian cocycles appear, this time in the context of $\mathbb{R}$-actions, is in time changes in flows. Suppose that φ_t is a flow. If $\gamma\colon M \to \mathbb{R}$, then the function $\alpha\colon \mathbb{R} \times M \to \mathbb{R}$ defined by

$$\alpha(t,x) = \int_0^t \gamma(\varphi_s(x)) \, ds$$

satisfies the cocycle condition:

$$\alpha(s+t,x) = \alpha(s,\varphi_t(x)) + \alpha(t,x), \tag{3}$$

which is the natural analogue of (1) for $\mathbb{R}$-actions. In general, if $\alpha\colon \mathbb{R} \times M \to \mathbb{R}$ is an arbitrary function, then the map $\psi^\alpha\colon \mathbb{R} \times M \to M$ given by

$$\psi^\alpha(t,x) = \varphi_{\alpha(t,x)}(x)$$

will define a flow on M if and only if α satisfies (3). Here too, one has a coboundary equation which corresponds to (2) for flows:

$$\alpha(t,x) - \beta(t,x) = \int_0^t \gamma(\varphi_s(x)) \, ds. \tag{4}$$

One can check that if Equation (2) is satisfied for cocycles α and β and some real-valued function γ, then the flows φ^α and φ^β are time changes of one another.

Linear cocycles. — By a *linear cocycle* we will mean a cocycle with values in a matrix group. Such non-abelian cocycles also arise naturally, most notably as derivative cocycles. Suppose that $f\colon M \to M$ is a diffeomorphism of an n-manifold M. To avoid technical issues, assume that the tangent bundle TM is trivial:

$$TM = M \times \mathbb{R}^d.$$

Then the derivative Df can be represented as a map $Df\colon M \to \mathrm{GL}(d,\mathbb{R})$ which, by the Chain Rule, satisfies the (non-abelian) cocycle condition:

$$D_x f^{n+m} = D_{f^m(x)} f^n \cdot D_x f^m.$$

(We remark that the case where TM is non-trivial can be handled with a slight generalization of the notion of cocycle, using sections of an appropriate bundle.) The group $\mathrm{GL}(d,\mathbb{R})$ can be replaced by other matrix groups, such as $\mathrm{SL}(d,\mathbb{R})$, $\mathrm{Sp}(d,\mathbb{R})$, $O(d)$, $U(d)$, etc. Such group-valued cocycles arise naturally as diffeomorphism cocycles that are volume preserving, symplectic, isometric, and so on, as well as in the study of frame flows on Riemannian manifolds.

Somewhat further afield, linear cocycles play a key role in analyzing the spectrum of the one-dimensional discrete Schrödinger operators. To any abelian cocycle ϕ over an ergodic system $f\colon M \to M$ and any $p \in M$ one can associate a one-dimensional discrete Schrödinger operator $H\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$H(x)_n = x_n + x_{n-1} - \phi(f^n(p))\, x_n.$$

The properties of the $SL(2,\mathbb{R})$-valued cocycles defined by

$$A_E(p) = \begin{pmatrix} E-\phi & -1 \\ 1 & 0 \end{pmatrix}$$

for different choices of the parameter $E \in \mathbb{R}$ determine the spectral properties of the operator H. For example, if this cocycle is uniformly hyperbolic for some value of E, then E lies in the resolvent set of H.

3. The central problems

We briefly outline the main questions that are addressed in the two papers in this volume.

Cohomological equation. — The *cohomological (or coboundary) equation* is

$$\phi = \Phi^{-1} \cdot (\Phi \circ f). \tag{5}$$

For abelian cocycles this is usually written:

$$\phi = \Phi \circ f - \Phi. \tag{6}$$

If such a solution exists, then ϕ is called a *coboundary.* Coboundaries are in a natural sense orthogonal to f-invariant functions: they are the image of the linear operator $\phi \mapsto \phi \circ f - \phi$, whereas the f-invariant functions are the kernel. This orthogonality