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SOME REGULARITIES AND SINGULARITIES APPEARING IN THE STUDY OF POLYNOMIALS AND OPERATORS

by

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Abstract. — We apply the viewpoint of singularity theory to the following problems: how does the decomposition of a polynomial P as the product of polynomials behave under perturbations of P? How do the eigenvalues, eigenspaces and more generally invariant subspaces of an operator A behave under perturbations of A? We give a characterization of the regular situations and describe completely the singular ones in some moderately degenerate situations.

Résumé **(Quelques regularités et singularités apparaissant dans l'étude des polynômes et des opérateurs)**

Nous appliquons le point de vue de la théorie des singularités aux deux problèmes suivants : comment la décomposition d'un polynôme P comme produit de polynômes se comporte-t-elle quand on perturbe P ? Comment les valeurs propres, vecteurs propres et plus généralement sous-espaces invariants d'un opérateur A se comportent-ils quand on perturbe A ? Nous caractérisons les situations régulières et décrivons complètement celles qui sont singulières mais pas trop dégénérées.

Introduction

In the study of bifurcations of dynamical systems one has to deal frequently with the following situation: as a parameter varies one considers the variation of an eigenvalue or of the invariant line generated by the corresponding eigenvector of the linearization of the dynamical system at a certain point. It often happens that those elements vary smoothly with the parameter, which is known to be the case if the eigenvalue is simple. But nevertheless the system undergoes a bifurcation if the eigenvalue crosses a certain subset of the plane (the unit circle, the imaginary axis, etc.). A second, more complex, situation happens when the eigenvalue becomes multiple, since then its variation with the parameter ceases to be smooth. The same situations occur when instead of an

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invariant line one needs to consider an invariant subspace of dimension greater than one.

During the years we have meditated on these questions and have arrived at various forms of expressing the (essentially known) conditions for the smooth variation of those elements (see for example **[5, 4]** for recent versions). One of those forms seems especially suited for studying, in terms of singularities of mappings, the situations where that variation ceases to be smooth. In this article we describe the simplest of those singularities.

The results. — We begin by a study of the simplest singularities of the polynomial multiplication map:

$$\mathrm{Mult} : \mathrm{MP}(n) \times \mathrm{MP}(m) \to \mathrm{MP}(n+m)$$

where $\mathrm{MP}(n)$ will denote the space of monic polynomials of degree n over $\mathbf{K}$, which will be either the real or the complex field. The rank of this map at a point (f, g) can be expressed in terms of the degree of the greatest common divisor $\gcd(f, g)$ so that it is a local diffeomorphism precisely when this degree is 0, i.e. when the factors are relatively prime. And we can describe completely the singularities of Mult when this degree is 1 (Theorem 1). Then we proceed to study the higher corank singularities of Mult; here our results are not as sharp, but we have a complete geometric description of many cases and an algebraic description of the rest.

As a byproduct of Theorem 1 we give an interesting description of the classical resultant of two polynomials and we obtain the relation between the singularities of Mult we describe and the resultant set $\mathrm{Res}(f, g) = 0$.

Then we apply Theorem 1 (and its corollary, Theorem 3, which generalizes it to the multiplication of an arbitrary number of factors) to study the singularities of the (monic) characteristic polynomial map

$$\chi : \mathrm{M}(n \times n) \to \mathrm{MP}(n)$$

where $\mathrm{M}(n \times n)$ denotes the space of $n \times n$ matrices with entries in $\mathbf{K}$. We will view each $M \in \mathrm{M}(n \times n)$ as a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and always take into account all its *complex* eigenvalues. We determine the matrices at which χ is a submersion and give a description of its simplest singularities (Theorem 5).

All the above is used to study the singularities of the eigenvalues of operators. For that, we introduce the set of all *proper elements* of a Banach space E over $\mathbf{K}$ to be the space of triples consisting of a linear operator on E, an invariant line and the corresponding eigenvalue:

$$\mathrm{Eig}(E) := \{(\lambda, L, A) \in \mathbf{K} \times \mathbf{P}(E) \times \mathrm{End}(E) \,:\, A(L) \subseteq L \textit{ and } A|_L = \lambda\}.$$

Here $\mathbf{P}(E)$ denotes the projectivization of E and $\mathrm{End}(E)$ the space of continuous linear endomorphisms of E. There is a natural projection $\Pi : \mathrm{Eig}(E) \mapsto \mathrm{End}(E)$ on the third factor.

The basic fact here (Theorem 6) is that $\mathrm{Eig}(E)$ is a *smooth object*, actually an analytic manifold modelled on $\mathrm{End}(E)$, provided with a projection Π onto $\mathrm{End}(E)$.

Therefore it is a kind a *resolution* of all the singularities associated to eigenvalue problems.

We show that, not surprisingly, Π is a local diffeomorphism precisely at those points where λ is a simple eigenvalue of A. And we can describe completely the singularities of Π when λ is a geometrically simple eigenvalue of A of finite multiplicity (Theorem 8). In the finite dimensional case this means simply that it has only one corresponding invariant line, while in infinite dimensions there are some technical additional conditions. We also show that the mapping that forgets the invariant line is regular (in this case an immersion) precisely when the eigenvalue is geometrically simple, a fact that is useful in the proof of the singularity part of Theorem 8.

In section D, we generalize this to invariant subspaces of dimension greater than one. In fact, this was the starting point of the whole story: in [**2**], we explained that the theory of formal normal forms for dynamical systems is an easy consequence of the Jordan decomposition of endomorphisms. Thinking about the generalization of this approach to *families*, we came to the conclusion that each characteristic space F_0 of an endomorphism A_0 of $\mathbf{C}^n$ must have the following stability property: every nearby endomorphism A has a unique invariant subspace $F(A)$ of the same dimension as $F_0 = F(A_0)$ and close to it, depending analytically on A. This is an easy result but it is not so well-known(1), and in section D we consider (and extend) it in the spirit of singularity theory.

The singularities. — The singularities found in Theorems 1, 8, 15, 16 are a certain type of Morin singularities which we will call *swallowtails*:

The *standard k–swallowtail* is the map

$$\mathrm{SW}_k : \mathbf{K}^{k-1} \to \mathbf{K}^{k-1}$$

defined by

$$\mathrm{SW}_k(a_1, \ldots, a_{k-2}, u) := (a_1, \ldots, a_{k-2}, u^k + a_{k-2}u^{k-2} + \cdots + a_1 u)$$

For us a *k–swallowtail* will be any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map of the form

$$\mathrm{SW}_k \times \mathrm{Id} : \mathbf{K}^{k-1} \times E \to \mathbf{K}^{k-1} \times E$$

for some Banach space E. When $\mathbf{K} = \mathbf{C}$ but E is *real*—a situation occurring whenever a real polynomial or endomorphism has nonreal roots or eigenvalues—we call such a map a *complex swallowtail.*

Interesting examples of k-swallowtails are the evaluation map

$$\begin{array}{rcl} \mathrm{ev} : \mathrm{MP}(k) \times \mathbf{K} & \to & \mathrm{MP}(k) \times \mathbf{K} \\ (P, a) & \mapsto & \big(P, P(a)\big) \end{array}$$

and the mapping

$$(a_1, \ldots, a_{k-1}, a) \mapsto (aa_1, a_1 + aa_2, a_2 + aa_3, \ldots, a_{k-2} + aa_{k-1}, a_{k-1} + a)$$

(1) The finite dimensional case led us to Theorem 1...

The second example shows that all swallowtails can be given by maps all of whose coordinate functions are polynomials of degree at most 2, a fact that we have not seen in the literature.

These examples, and some of their variants, will play an important role in the proofs of the theorems.

The singularities in Theorem 5 will be *k–swallowtail deformations*, by which we mean any map germ between two Banach spaces which is diffeomorphic to the germ at 0 of a map

$$G : E \times E' \to E$$

such that $G(x, 0)$ is a k–swallowtail, where E, E' are Banach spaces.

There are many k–swallowtail deformations between spaces of the same dimension, so this term does not describe a precise singularity type. And though it is possible, in principle, to describe them all, there remains to do so specifically for the singularities of χ.

We will show by examples that in all cases the singularities that are not swallowtails are more complicated than those one could expect from the classification results of singularities of mappings.

We hope to give soon some applications of these results to bifurcation problems of dynamical systems.

In the Appendices we recall the main properties of the singularities we will use in the text and describe completely the main examples. We also provide an introduction to the results on continuous linear maps between Banach spaces needed in the text, referring to Rudin's beautiful book **[10]** for more details about this magnificent theory.

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A. Singularities of Polynomial Multiplication

Polynomial multiplication defines a map

$$\mathrm{Mult} : \mathrm{MP}(n) \times \mathrm{MP}(m) \to \mathrm{MP}(n+m)$$

We are interested in describing the regular points and the singularities of the map Mult. We will denote by $\gcd(P, Q)$ the monic greatest common divisor of the monic polynomials P and Q.

Theorem 1. — *For (P_0, Q_0) in $\mathrm{MP}(n) \times \mathrm{MP}(m)$,*

(i) *The corank of the differential $D\,\mathrm{Mult}(P_0, Q_0)$ is the degree of $\gcd(P_0, Q_0)$.*
(ii) *In particular,* Mult *is a local diffeomorphism at (P_0, Q_0) if and only if $\gcd(P_0, Q_0) = 1$.*
(iii) *The mapping* Mult *is a $(k+1)$–swallowtail at (P_0, Q_0) for some positive integer k if, and only if, $\deg \gcd(P_0, Q_0) = 1$, the integer k being the maximum of the multiplicities in P_0 and Q_0 of their common root.*

(iv) *If* $\mathbf{K} = \mathbf{R}$*, the mapping* Mult *is a complex* $(k+1)$*–swallowtail at* (P_0, Q_0) *for some positive integer* k *if, and only if,* $\gcd(P_0, Q_0)$ *is an irreducible polynomial of degree* 2*, the integer* k *being the maximum of the multiplicities in* P_0 *and* Q_0 *of their complex conjugate common roots*[2]*.*

Proof. — The tangent space of $\mathrm{MP}(n)$ at any point is the set of polynomials of degree less that n. The derivative of Mult at (P_0, Q_0) is then given by

$$(P, Q) \mapsto P_0 Q + P Q_0.$$

Therefore its image, being the set of multiples of $\gcd(P_0, Q_0)$ by polynomials of degree less than $n + m - \deg \gcd(P_0, Q_0)$, has this dimension. This proves (i) and therefore (ii).

If $\gcd(P_0, Q_0) = x - \alpha$ then $x - \alpha$ must divide one of P_0, Q_0 with multiplicity 1 and the other one with multiplicity k. By changing the variable in the polynomials (which induces a diffeomorphism of $\mathrm{MP}(n)$) we can assume $\alpha = 0$.

Consider first the case $P_0 = x$, $Q_0 = x^k$. Then Mult is given by

$$\mathrm{Mult}\left(x + a, x^k + \sum_{i=0}^{k-1} a_i x^i\right) = \sum_{i=0}^{k+1} (a_{i-1} + a a_i) x^i$$

(putting $a_k = 1$, $a_{k+1} = a_{-1} = 0$) or, in coordinates $(a, a_{k-1}, \dots, a_0)$, by

$$\mathrm{Mult}(a_0, \dots, a_{k-1}, a) = (a a_0, a_0 + a a_1, a_1 + a a_2, \dots, a_{k-1} + a),$$

which is a $(k+1)$–swallowtail by example 2 in Appendix A.

In general, let $P_0 = x P_1$, $Q_0 = x^k Q_1$, where P_1, Q_1 are not divisible by x. Then, setting $m_1 := m - k$ and $n_1 := n - 1$, we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{MP}(n) & \times & \mathrm{MP}(m) & \to & \mathrm{MP}(m+n) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{MP}(1) \times \mathrm{MP}(n_1) & \times & \mathrm{MP}(k) \times \mathrm{MP}(m_1) & \to & \mathrm{MP}(k+1) \times \mathrm{MP}(m_1 + n_1)
\end{array}$$

where all maps are given by multiplication. By Theorem 1, the vertical arrows are local diffeomorphisms at (x, P_1), (x^k, Q_1) and $(x^{k+1}, P_1 Q_1)$ respectively. The lower map is the product of the multiplication $\mathrm{MP}(1) \times \mathrm{MP}(k) \to \mathrm{MP}(k+1)$, which we have just seen to be a $(k+1)$–swallowtail at (x, x^k), and the multiplication $\mathrm{MP}(n_1) \times \mathrm{MP}(m_1) \to \mathrm{MP}(m_1 + n_1)$, a local diffeomorphism at (P_1, Q_1) b Theorem 1. Therefore the upper multiplication map is diffeomorphic to the lower one, which is a $(k+1)$–swallowtail. This proves the "if" in (i). As for the "only if", just notice that, when the degree of $\gcd(P_0, Q_0)$ is greater than 1, the corank of $D\,\mathrm{Mult}$ is greater than 1 and Mult cannot be a swallowtail at that point. This proves (iii).

[2] Or, in other words, the greatest integer k such that $\gcd(P_0, Q_0)^k$ divides P_0 or Q_0.