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HOLONOMY INVARIANCE: ROUGH REGULARITY AND APPLICATIONS TO LYAPUNOV EXPONENTS

by

Artur Avila, Jimmy Santamaria & Marcelo Viana

Abstract. — Un cocycle lisse est un produit gauche qui agit par des difféomorphismes dans les fibres. Si les exposants de Lyapounov extremaux du cocycle coincident alors les fibres possèdent certaines structures qui sont invariantes, à la fois, par la dynamique et par un pseudo-groupe canonique de transformations d'holonomie. Nous démontrons ce *principe d'invariance* pour les cocycles lisses au dessus des difféomorphismes conservatifs partiellement hyperboliques, et nous en donnons des applications aux cocycles linéaires et aux dynamiques partiellement hyperboliques.

Résumé. — Skew-products that act by diffeomorphisms on the fibers are called smooth cocycles. If the extremal Lyapunov exponents of a smooth cocycle coincide then the fibers carry quite a lot of structure that is invariant under the dynamics and under a canonical pseudo-group of holonomy maps. We state and prove this *invariance principle* for cocycles over partially hyperbolic volume preserving diffeomorphisms. It has several applications, e.g., to linear cocycles and to partially hyperbolic dynamics.

1. Introduction

Lyapunov exponents measure the asymptotic rates of contraction and expansion, in different directions, of smooth dynamical systems such as diffeomorphisms, cocycles, or their continuous-time counterparts. These numbers are well defined on a full measure subset of phase-space, relative to any finite invariant measure. Systems whose Lyapunov exponents are distinct/non-vanishing exhibit a wealth of geometric and dynamical structure (invariant laminations, entropy formula, abundance of periodic orbits, dimension of invariant measures) on which one can build to describe their evolution. The main theme we are interested in is that systems for which the Lyapunov exponents are *not* distinct are also special, in that they satisfy a very strong invariance principle. Thus, a detailed theory can be achieved also in this case, if only using very different ingredients.

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In the special case of linear systems, the invariance principle can be traced back to the classical results on random matrices by Furstenberg [**12**], Ledrappier [**19**], and others. Moreover, it has been refined in more recent works by Bonatti, Gomez-Mont, Viana [**7**], Bonatti, Viana [**8**], Viana [**25**] and Avila, Viana [**1, 2**]. An explicit and much more general formulation, that applies to smooth (possibly non-linear) systems, is proposed in Avila, Viana [**3**] and the present paper: while [**3**] deals with extensions of hyperbolic transformations, here we handle the case when the base dynamics is just partially hyperbolic and volume preserving. The two papers are contemporary and closely related: in particular, Theorem A of [**3**] relies on a version of the invariance principle proved in here, more precisely, Theorem B below.

As an illustration of the reach of our methods, let us state the following application in the realm of partially hyperbolic dynamics (for details, see Remark 2.9). Let $f : M \to M$ be a C^2 partially hyperbolic, dynamically coherent, volume preserving, accessible diffeomorphism satisfying a suitable center bunching condition. If the center bundle E^c has dimension 2 and the center Lyapunov exponents coincide almost everywhere then f admits

(a) either an invariant continuous field of directions $r \subset E^c$,
(b) or an invariant continuous field of pairs of directions $r_1 \cup r_2 \subset E^c$,
(c) or an invariant continuous conformal structure on E^c.

Sometimes, one can exclude all three alternatives a priori. That is the case, for instance, if f is known to have periodic points p and q that are, respectively, elliptic and hyperbolic along the center bundle E^c, in the following sense: the center eigenvalues of p are neither real nor pure imaginary, and the center eigenvalues of q are real and distinct. Then it follows that the center Lyapunov exponents are distinct and, in particular, at least one is non-zero. If f is symplectic then both center Lyapunov exponents are different from zero; compare Theorem A in [**3**].

Precise statements of our results, including the definitions of the objects involved, will appear in the next section. Right now, let us observe that important applications of the methods developed in here have been obtained by several authors: a Livšic theory of partially hyperbolic diffeomorphism, by Wilkinson [**27**]; existence and properties of physical measures, by Viana, Yang [**26**]; construction of measures of maximal entropy, by Hertz, Hertz, Tahzibi, Ures [**22**].

2. Preliminaries and statements

2.1. Partially hyperbolic diffeomorphisms. — Throughout the paper, unless stated otherwise, $f : M \to M$ is a partially hyperbolic diffeomorphism on a compact manifold M and μ is a probability measure in the Lebesgue class of M. In this section we define these and other related notions. See [**9, 15, 16, 24**] for more information.

A diffeomorphism $f : M \to M$ of a compact manifold M is *partially hyperbolic* if there exists a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u \tag{2.1}$$

invariant under the derivative Df, a Riemannian metric $\|\cdot\|$ on M, and positive continuous functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with $\nu, \hat{\nu} < 1$ and $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ such that, for any unit vector $v \in T_pM$,

$$\|Df(p)v\| < \nu(p) \qquad \text{if } v \in E^s(p), \tag{2.2}$$

$$\gamma(p) < \|Df(p)v\| < \hat{\gamma}(p)^{-1} \qquad \text{if } v \in E^c(p), \tag{2.3}$$

$$\hat{\nu}(p)^{-1} < \|Df(p)v\| \qquad \text{if } v \in E^u(p). \tag{2.4}$$

(Equivalently, one could ask these conditions for some iterate; see Gourmelon [**14**].) All three subbundles E^s, E^c, E^u are assumed to have positive dimension. However, in some cases (cf. Remarks 3.12 and 4.2) one may let either $\dim E^s = 0$ or $\dim E^u = 0$.

We take M to be endowed with the distance dist associated to such a Riemannian structure. The *Lebesgue class* is the measure class of the volume induced by this (or any other) Riemannian metric on M. These notions extend to any submanifold of M, just considering the restriction of the Riemannian metric to the submanifold. We say that f is *volume preserving* if it preserves some probability measure in the Lebesgue class of M.

Suppose that $f : M \to M$ is partially hyperbolic. The stable and unstable bundles E^s and E^u are uniquely integrable and their integral manifolds form two transverse continuous foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed submanifolds of the same class of differentiability as f. These foliations are referred to as the *strong-stable* and *strong-unstable* foliations. They are invariant under f, in the sense that

$$f(\mathcal{W}^s(x)) = \mathcal{W}^s(f(x)) \quad \text{and} \quad f(\mathcal{W}^u(x)) = \mathcal{W}^u(f(x)),$$

where $\mathcal{W}^s(x)$ and $\mathcal{W}^s(x)$ denote the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, passing through any $x \in M$. These foliations are, usually, *not* transversely smooth: the holonomy maps between any pair of cross-sections are not even Lipschitz continuous, in general, although they are always γ-Hölder continuous for some $\gamma > 0$. Moreover, if f is C^2 then these foliations are absolutely continuous, meaning that the holonomy maps preserve the class of zero Lebesgue measure sets. Let us explain this key fact more precisely.

Let $d = \dim M$ and $\mathcal{F}$ be a continuous foliation of M with k-dimensional smooth leaves, $0 < k < d$. Let $\mathcal{F}(p)$ be the leaf through a point $p \in M$ and $\mathcal{F}(p, R) \subset \mathcal{F}(p)$ be the neighborhood of radius $R > 0$ around p, relative to the distance defined by the Riemannian metric restricted to $\mathcal{F}(p)$. A *foliation box* for $\mathcal{F}$ at p is the image of an embedding

$$\Phi : \mathcal{F}(p, R) \times \mathbb{R}^{d-k} \to M$$

such that $\Phi(\cdot, 0) = \mathrm{id}$, every $\Phi(\cdot, y)$ is a diffeomorphism from $\mathcal{F}(p, R)$ to some subset of a leaf of $\mathcal{F}$ (we call the image a *horizontal slice*), and these diffeomorphisms vary continuously with $y \in \mathbb{R}^{d-k}$. Foliation boxes exist at every $p \in M$, by definition of continuous foliation with smooth leaves. A *cross-section* to $\mathcal{F}$ is a smooth codimension-k disk inside a foliation box that intersects each horizontal slice exactly once, transversely and with angle uniformly bounded from zero.

Then, for any pair of cross-sections Σ and Σ', there is a well defined *holonomy map* $\Sigma \to \Sigma'$, assigning to each $x \in \Sigma$ the unique point of intersection of Σ' with the horizontal slice through x. The foliation is *absolutely continuous* if all these homeomorphisms map zero Lebesgue measure sets to zero Lebesgue measure sets. That holds, in particular, for the strong-stable and strong-unstable foliations of partially hyperbolic C^2 diffeomorphisms and, in fact, the Jacobians of all holonomy maps are bounded by a uniform constant.

A measurable subset of M is *s-saturated* (or $\mathcal{W}^s$*-saturated*) if it is a union of entire strong-stable leaves, *u-saturated* (or $\mathcal{W}^u$-saturated) if it is a union of entire strong-unstable leaves, and *bi-saturated* if it is both s-saturated and u-saturated. We say that f is *accessible* if $\varnothing$ and M are the only bi-saturated sets, and *essentially accessible* if every bi-saturated set has either zero or full measure, relative to any probability measure in the Lebesgue class. A measurable set $X \subset M$ is *essentially s-saturated* if there exists an s-saturated set $X^s \subset M$ such that $X \Delta X^s$ has measure zero, for any probability measure in the Lebesgue class. *Essentially u-saturated* sets are defined analogously. Moreover, X is *bi-essentially saturated* if it is both essentially s-saturated and essentially u-saturated.

Pugh, Shub conjectured in **[20]** that essential accessibility implies ergodicity, for a C^2 partially hyperbolic, volume preserving diffeomorphism. In **[21]** they showed that this does hold under a few additional assumptions, called dynamical coherence and center bunching. To date, the best result in this direction is due to Burns, Wilkinson **[10]**, who proved the Pugh-Shub conjecture assuming only the following mild form of center bunching:

Definition 2.1. — A C^2 partially hyperbolic diffeomorphism is *center bunched* if the functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ in (2.2)–(2.4) may be chosen to satisfy

$$\nu < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma\hat{\gamma}. \tag{2.5}$$

When the diffeomorphism is just $C^{1+\alpha}$, for some $\alpha > 0$, the arguments of Burns, Wilkinson **[10]** can still be carried out, as long as one assumes what they call strong center bunching (see [**10**, Theorem 0.3]). All our results extend to this setting.

2.2. Fiber bundles. — In this paper we deal with a few different types of fiber bundles over the manifold M. The more general type we consider are *continuous fiber bundles* $\pi : \mathcal{E} \to M$ modeled on some topological space N. By this we mean that $\mathcal{E}$ is a topological space and there is a family of homeomorphisms (*local charts*)

$$\phi_U : U \times N \to \pi^{-1}(U), \tag{2.6}$$

indexed by the elements U of some finite open cover $\mathcal{U}$ of M, such that $\pi \circ \phi_U$ is the canonical projection $U \times N \to U$ for every $U \in \mathcal{U}$. Then each $\phi_{U,x} : \xi \mapsto \phi_U(x, \xi)$ is a homeomorphism between N and the *fiber* $\mathcal{E}_x = \pi^{-1}(x)$.

An important role will be played by the class of *fiber bundles with smooth fibers*, that is, continuous fiber bundles whose fibers are manifolds endowed with a continuous Riemannian metric. More precisely, take N to be a Riemannian manifold, not necessarily complete, and assume that all coordinate changes $\phi_V^{-1} \circ \phi_U$ have the form

$$\phi_V^{-1} \circ \phi_U : (U \cap V) \times N \to (U \cap V) \times N, \quad (x, \xi) \mapsto (x, g_x(\xi)) \tag{2.7}$$

where:

(i) $g_x : N \to N$ is a C^1 diffeomorphism and the map $x \mapsto g_x$ is continuous, relative to the uniform C^1 distance on $\mathrm{Diff}^1(N)$ (the uniform C^1 distance is defined by $\mathrm{dist}_{C^1}(g_x, g_y) = \sup\{|g_x(\xi) - g_y(\xi)|, \|Dg_x(\xi) - Dg_y(\xi)\| : \xi \in N\}$);

(ii) the derivatives $Dg_x(\xi)$ are $Dg_x^{-1}(\xi)$ are uniformly continuous and uniformly bounded in norm.

Endow each $\mathcal{E}_x$ with the manifold structure that makes $\phi_{U,x}$ a diffeomorphism. Condition (i) ensures that this does not depend on the choice of $U \in \mathcal{U}$ containing x. Moreover, consider on each $\mathcal{E}_x$ the Riemannian metric $\gamma_x = \sum_{U \in \mathcal{U}} \rho_U(x)\gamma_{U,x}$, where $\gamma_{U,x}$ is the Riemannian metric transported from N by the diffeomorphism $\phi_{U,x}$ and $\{\rho_U : U \in \mathcal{U}\}$ is a partition of unit subordinate to $\mathcal{U}$. It is clear that γ_x depends continuously on x. Condition (ii) ensures that different choices of the partition of unit give rise to Riemannian metrics γ_x that differ by a bounded factor only.

Restricting even further, we call $\pi : \mathcal{E} \to M$ a *continuous vector bundle* of dimension $d \geq 1$ if $N = \mathbb{K}^d$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and every g_x is a linear isomorphism, depending continuously on x and such that $\|g_x^{\pm 1}\|$ are uniformly bounded. Then each fiber $\mathcal{E}_x$ is isomorphic to $\mathbb{K}^d$ and is equipped with a scalar product (and, hence, a norm) which is canonical up to a bounded factor.

We also need to consider more regular vector bundles. Given $r \in \{0, 1, \ldots, k, \ldots\}$ and $\alpha \in [0, 1]$, we say that $\pi : \mathcal{E} \to M$ is a $C^{r,\alpha}$ vector bundle if, for any U, $V \in \mathcal{U}$ with non-empty intersection, the map

$$U \cap V \to \mathrm{GL}(d, \mathbb{K}), \quad x \mapsto g_x \tag{2.8}$$

is of class $C^{r,\alpha}$, that is, it is r times differentiable and the derivative of order r is α-Hölder continuous.

2.3. Linear cocycles. — Let $\pi : \mathcal{V} \to M$ be a continuous vector bundle of dimension $d \geq 1$. A *linear cocycle* over $f : M \to M$ is a continuous transformation $F : \mathcal{V} \to \mathcal{V}$ satisfying $\pi \circ F = f \circ \pi$ and acting by linear isomorphisms $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$ on the fibers. By Furstenberg, Kesten [**13**], the *extremal Lyapunov exponents*

$$\lambda_+(F, x) = \lim_{n \to \infty} \frac{1}{n} \log \|F_x^n\| \quad \text{and} \quad \lambda_-(F, x) = \lim_{n \to \infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1}$$

exist at μ-almost every $x \in M$, relative to any f-invariant probability measure μ. If (f, μ) is ergodic then they are constant on a full μ-measure set. It is clear that $\lambda_-(F, x) \leq \lambda_+(F, x)$ whenever they are defined. We study conditions under which these two numbers coincide.

Suppose that $\pi : \mathcal{V} \to M$ is a $C^{r,\alpha}$ vector bundle, for some fixed r and α, and f is also of class $C^{r,\alpha}$ (this is contained in our standing assumptions if $r + \alpha \leq 2$). Then