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INFINITE DIMENSIONAL OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE FUNCTION AND THE TRACE FORMULA FOR THE HEAT SEMIGROUP

by

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It is a special honour and pleasure to dedicate this work
to Jean-Michel Bismut, as a small sign of gratitude
for all he has taught us by his inspiring work

Abstract. — Infinite dimensional oscillatory integrals with a polynomially growing phase function with a small parameter $\epsilon \in \mathbb{R}^+$ are studied by means of an analytic continuation technique, as well as their asymptotic expansion in the limit $\epsilon \downarrow 0$. The results are applied to the study of the semiclassical behavior of the trace of the heat semigroup with a polynomial potential.

***Résumé* (Intégrales oscillantes en dimension infinie avec une phase polynomiale et formule de la trace pour le semigroupe de la chaleur)**
Nous étudions les intégrales oscillantes en dimension infinie avec une phase de croissance polynomiale à petit paramètre $\epsilon \in \mathbb{R}^+$ au moyen d'une technique de prolongement analytique. Nous donnons aussi leur développement asymptotique en ϵ lorsque $\epsilon \downarrow 0$. Nous présentons une application de ces résultats à l'étude du comportement semiclassique de la trace du noyau de la chaleur avec un potentiel polynomial.

1. Introduction

Oscillatory integrals on finite dimensional Hilbert spaces, i.e. expressions of the form

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\epsilon}\Phi(x)} g(x)dx, \tag{1}$$

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(where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the phase function and $\epsilon \in \mathbb{R}^+$ a real positive parameter) are a classical topic of investigation, having several applications, e.g. in electromagnetism, optics and acoustics. They are part of the general theory of Fourier integral operators **[27, 35]**. Particularly interesting is the study of the asymptotic behavior of these integrals in the limit $\epsilon \downarrow 0$. The generalization of the definition of oscillatory integrals to the case where the integration is performed on an infinite dimensional space, in particular a space of continuous functions, presents a particular interest in connection with applications to quantum theory such as the mathematical realization of Feynman path integrals **[1, 7]** (see also, e.g. **[26, 36]** and references therein; applications include—besides quantum mechanics—quantum field theory and low dimensional geometry, see, e.g. **[10]** and references therein). In the case where the integration is performed on such spaces and on general real separable Hilbert spaces, the theory was for a long time restricted to oscillatory integrals with phase functions Φ which can be written as sums of a quadratic form and a bounded function belonging to the class of Fourier transforms of complex measures. In **[8, 9]** these results have been generalized to phase functions with quartic polynomial growth. In this paper we consider a generalization of the oscillatory integral (1) and its infinite dimensional analogue, in the case where the imaginary unity i in the exponent is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int e^{-\frac{s}{\epsilon}\Phi(x)} g(x)dx. \tag{2}$$

Strictly speaking $I(s)$ has an oscillatory behavior only for s being a pure imaginary number. By generalizing the results of **[8]**, we prove (in section 2) a representation formula which allows us to compute an infinite dimensional oscillatory integral of the form (2), with a phase function Φ having an arbitrary even polynomial growth, in terms of a Gaussian integral. In the non degenerate case (i.e. when the Hessian of the phase function is non degenerate), we compute (in section 3) the asymptotic expansion of the integral as $\epsilon \downarrow 0$ in powers of ϵ. In the degenerate case the situation is more involved. In section 4 we handle in detail a particular example and apply this result to the study of the asymptotic behavior of the trace of the heat semigroup $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, in the case where H is the essentially self-adjoint operator on $C_0^\infty \equiv C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ given on the functions $\phi \in C_0^\infty$ by

$$H\phi(x) = \Big(-\frac{\hbar^2}{2}\Delta_x + V(x)\Big)\phi(x), \tag{3}$$

where $\hbar > 0$ and V is a polynomially growing potential of the form $V(x) = |x|^{2N}$, $x \in \mathbb{R}^d$, $N \in \mathbb{N}$. This corresponds to exhibiting the detailed behavior of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, "near the classical limit". Indeed H can be interpreted as a Schrödinger Hamiltonian (in which case $\hbar$ is the reduced Planck's constant), and consequently $e^{-\frac{t}{\hbar}H}$

as a Schrödinger semigroup with imaginary time, i.e. the heat semigroup. In recent years a particular interest has been devoted to the study of the trace of the heat semigroup and of the corresponding Schrödinger group $e^{-\frac{it}{\hbar}H}$, $t \in \mathbb{R}$, (related to the heat semigroup by analytic continuation in the "time variable" t) and their asymptotics in the "semiclassical limit $\hbar \downarrow 0$" (see, e.g., [**46**], [**1, 4, 12**] and also [**16, 17, 18, 20**] for related problems). In particular one is interested in the proof of a trace formula of Gutzwiller's type, relating the asymptotics of the trace of the Schrödinger group and the spectrum of the quantum mechanical energy operator H with the classical periodic orbits of the system. Gutzwiller's heuristic trace formula, which is a basis of the theory of quantum chaotic systems, is the quantum mechanical analogue of Selberg's trace formula, relating the spectrum of the Laplace-Beltrami operator on manifolds with constant negative curvature with the periodic geodesics (see, e.g., [**25**] and [**3, 4, 12**]).
In the case where the potential V is the sum of an harmonic oscillator part and a bounded perturbation V_0 that is the Fourier transform of a complex (bounded variation) measure on $\mathbb{R}^d$, rigorous results on the asymptotics of the trace of the Schrödinger group and the heat semigroup have been obtained in [**4, 12**] by means of an infinite dimensional version of the stationary phase method for infinite dimensional oscillatory integrals (see [**7**] for a review of this topic).
The paper is organized as follows. In section 2 we give the definition and the main results on infinite dimensional oscillatory integrals of the form (2) with a polynomial phase function Φ, in section 3 we study the asymptotic expansion of the integral in the case where the origin is a non degenerate critical point of Φ, while in section 4 we study a degenerate case and apply these results to the asymptotics of $\mathrm{Tr}[e^{-\frac{t}{\hbar}H}]$, $t > 0$, as $\hbar \downarrow 0$.

2. Infinite dimensional oscillatory integrals

The present section is devoted to the study of the *oscillatory integrals with complex parameter s*. In the following we shall denote by $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ a real separable infinite dimensional Hilbert space, s will be a complex number such that $\mathrm{Re}(s) \geq 0$, $g : \mathcal{H} \to \mathbb{C}$ a Borel function.
Let us consider the generalization of the oscillatory integral (1) to the case (2) where the imaginary unity i in the exponent is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C}\ :\ \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int_{\mathbb{R}^n} e^{-\frac{s}{\epsilon}\Phi(x)} g(x) dx. \tag{4}$$

In the case where s is a pure imaginary number, by exploiting the oscillatory behavior of the integrand, the oscillatory integral (4) can still be defined as an improper Riemann integral even if the (continuous) function g is not summable. In the case where the phase function Φ is a quadratic form, the integral (4) is called *Fresnel integral*. We propose here for the general case (4) a modification of the Hörmander's definition [**27**], also considered in [**5, 23**] in connection to the generalization to the infinite dimensional case. This modification is as follows:

Definition 2.1. — Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel function, $s \in \mathbb{C}^+$ a complex parameter. Let $\mathcal{S}$ be a subset of the space of the Schwartz test functions $S(\mathbb{R}^n)$. If for each $\phi \in \mathcal{S}$ such that $\phi(0) = 1$ the integrals

$$I_\delta(f,\phi) := \int_{\mathbb{R}^n} (2\pi s^{-1})^{-n/2} e^{-\frac{s}{2}|x|^2} f(x)\phi(\delta x)dx$$

exist for all $\delta > 0$ and $\lim_{\delta\to 0} I_\delta(f,\phi)$ exist and is independent of ϕ, then this limit is called the Fresnel integral of f with parameter s (with respect to the space $\mathcal{S}$ of regularizing functions) and denoted by

$$\mathcal{F}^s(f) \equiv \widetilde{\int_{\mathbb{R}^n}^s} e^{-\frac{s}{2}|x|^2} f(x)dx. \tag{5}$$

By an adaptation of the definition of infinite dimensional oscillatory integrals given in [**23**] it is possible to define the *oscillatory integral with parameter s* on the Hilbert space $\mathcal{H}$, namely

$$I(s) = \widetilde{\int_{\mathcal{H}}^s} e^{-\frac{s}{2}\|x\|^2} g(x)dx \tag{6}$$

as the limit of a sequence of (suitably normalized) finite dimensional approximations [**12**].

Definition 2.2. — A Borel measurable function $f : \mathcal{H} \to \mathbb{C}$ is called $\mathcal{F}^s$ integrable if for each sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \le P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$ (I being the identity operator in $\mathcal{H}$), the finite dimensional approximations of the Fresnel integral of f, with parameter s,

$$\mathcal{F}^s_{P_n}(f) \equiv \widetilde{\int_{P_n\mathcal{H}}^s} e^{-\frac{s}{2}\|P_nx\|^2} f(P_nx)d(P_nx) \tag{7}$$

exist (in the sense of definition 2.1) and the limit $\lim_{n\to\infty} \mathcal{F}^s_{P_n}(g)$ exists and is independent of the sequence $\{P_n\}$.
In this case the limit is called the infinite dimensional Fresnel integral of f with parameter s and is denoted by

$$\widetilde{\int_{\mathcal{H}}^s} e^{-\frac{s}{2}\|x\|^2} f(x)dx.$$

f is then said to be integrable (in the sense of Fresnel integrals with parameter s).

The description of the largest class of functions which are integrable in this sense is an open problem, even in the finite dimensional case. Clearly it depends on the class $\mathcal{S}$ of the regularizations. The common choice is $\mathcal{S} \equiv S(\mathbb{R}^n)$, **[5, 23]**. In this case **[5, 7, 23]** the space of integrable functions includes (in finite as well as in infinite dimensions) the Fresnel class $\mathcal{F}(\mathcal{H})$, that is the set of functions $f : \mathcal{H} \to \mathbb{C}$ that are Fourier transforms of complex bounded variation measures on $\mathcal{H}$:

$$f(x) = \int_{\mathcal{H}} e^{i\langle y,x\rangle} d\mu_f(y) \equiv \hat{\mu}_f(x), \qquad x \in \mathcal{H}$$

$$\sup \sum_i |\mu_f(E_i)| < \infty,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$.
In fact for any $f \in \mathcal{F}(\mathcal{H})$ it is possible to prove a Parseval type equality that allows to compute the infinite dimensional oscillatory integral of f (with purely imaginary parameter s) in terms of an absolutely convergent integral with respect to the associated complex-valued measure μ_f **[5, 23]**. Indeed given a self-adjoint trace-class operator $B : \mathcal{H} \to \mathcal{H}$, such that $(I - B)$ is invertible, a function $f \in \mathcal{F}(\mathcal{H})$, $f = \hat{\mu}_f$ and a positive parameter $\hbar \in \mathbb{R}^+$, it is possible to prove that the function $e^{-\frac{i}{2\hbar}\langle x,Bx\rangle} f(x)$ is Fresnel integrable and the corresponding Fresnel integral with parameter $s = -i/\hbar$ is given by

$$(8) \quad \widetilde{\int_{\mathcal{H}}^{-i/\hbar}} e^{\frac{i}{2\hbar}\|x\|^2} e^{-\frac{i}{2\hbar}\langle x,Bx\rangle} e^{i\langle x,y\rangle} f(x)dx$$
$$= (\det(I - B))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha+y,(I-B)^{-1}(\alpha+y)\rangle} \mu_f(d\alpha)$$

where $\det(I - B) = |\det(I - B)|e^{-\pi i \text{ Ind } (I-B)}$ is the Fredholm determinant of the operator $(I - B)$, $|\det(I - B)|$ its absolute value and $\text{Ind}((I - B))$ is the number of negative eigenvalues of the operator $(I - B)$, counted with their multiplicities.

Let us also recall, for later use, a known result on infinite dimensional oscillatory integrals.

Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. Let also $\|\cdot\|$ be an equivalent norm on $\mathcal{H}$ with scalar product denoted by $\langle\cdot,\cdot\rangle$. Let us denote the new Hilbert space by $\tilde{\mathcal{H}}$. Let us assume moreover that

$$\langle x_1, x_2\rangle = (x_1, x_2) + (x_1, Tx_2), \qquad x_1, x_2 \in \tilde{\mathcal{H}}$$

$$\|x\|^2 = |x|^2 + (x, Tx), \qquad x \in \tilde{\mathcal{H}},$$