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HIDDEN STRUCTURES ON SEMISTABLE CURVES

by

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Abstract. — Let V be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let X be a proper curve over V with semistable special fiber and smooth generic fiber. In this article we explicitly describe the Frobenius and monodromy operators on the log crystalline cohomology of X with values in a regular log F-isocrystal in terms of p-adic integration. We have a version for open curves and as an application we prove that two differently defined $\mathscr{L}$-invariants, attached to a split multiplicative at p new elliptic eigenform, are equal.

***Résumé* (Structures cachées sur les courbes semi-stables).** — Soit V l'anneau des entiers d'une extension finie de $\mathbb{Q}_p$ et soit X une courbe propre sur V à fibre spéciale semistable et à fibre générique lisse. Dans cet article nous décrivons explicitement les opérateurs de Frobenius et de monodromie sur la cohomologie log cristalline de X à valeurs dans un log F-isocristal régulier, en termes d'intégration p-adique. Nous proposons une version pour les courbes ouvertes et en guise d'application nous prouvons que deux $\mathscr{L}$-invariants définis de façon différente, attachés à une forme modulaire nouvelle multiplicative en p, sont égaux.

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$ and X an algebraic variety over K. As Illusie remarked in *Cohomologie de de Rham et cohomologie étale p-adique* [I], "le groupe $H^1_{dR}(X/K)$ se trouve muni d'une structure plus riche qu'il n'y paraît de prime abord." This "hidden structure" has been discussed by many people including Berthelot and Ogus [BO] when X is proper with good reduction and more generally by Hyodo and Kato [HK]. In this paper, we expose it in the relative situation over a curve with semistable reduction using residues and p-adic integration. More precisely we study de Rham cohomology of a semi-stable curve with coefficients in the relative cohomology of a smooth proper family over that curve. The information on crystalline and de

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Rham cohomology of a curve with semi-stable reduction supplied by this article is similar to that of the theory of vanishing cycles for ℓ-adic cohomology.

Suppose K has residue field k and ring of integers V. Let $W := W(k)$ denote the ring of Witt-vectors with coefficients in k, K_0 its fraction field and we denote by σ the Frobenius automorphism of K_0. Let C_K be a smooth projective curve over K with a semi-stable model C over V. By this we mean that locally C is smooth over $\mathrm{Spec}(V)$ or étale over $\mathrm{Spec}\,(V[X,Y]/(XY-\pi))$, where π is a uniformizer of V. Denote by $\overline{C} := C \times_{\mathrm{Spec(V)}} \mathrm{Spec}(k)$, its special fiber and by Sing, the singular sub-scheme of $\overline{C}$.

Then the vector space $H^1_{dR}(C_K)$ has enough hidden structure so that one can recover the corresponding representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ on the étale cohomology of $C_{\overline{K}}$, à la Fontaine. I.e. besides the Hodge filtration it has a K_0-lattice (the log-crystalline cohomology of $\overline{C}$ with $\mathbb{Q}_p$-coefficients) with linear monodromy and σ-semi-linear Frobenius operators. One can use this to describe the representation. This is true much more generally (see for example **[18]** and **[39]**.)

Let $g\colon Z \longrightarrow C$ be a flat proper morphism. Suppose P is a sub-scheme of C, finite and étale over V whose reduction is disjoint from Sing. Let $C^\times$ be the log formal scheme over V associated to the pair (C,P) (i.e. the formal completion of C along its special fiber together with the log-structure associated to P). Denote $g^{-1}(P)$ by D_P and let $Z^\times$ be the log formal scheme over V associated to the pair (Z, D_P). We'll abuse notation and also let $g\colon Z^\times \longrightarrow C^\times$ denote the morphism of log formal schemes induced by g. Then D_P is a divisor of Z and we will suppose from now on that $D_P \cup \overline{Z}$ is a reduced divisor with normal crossings. Here $\overline{Z}$ is the special fiber of Z. Suppose that the restriction of g induces a smooth proper map $(Z\backslash D_P) \longrightarrow (C\backslash P)$. Then, under all of the assumptions above $g\colon Z^\times \to C^\times$ is log smooth.

For example, if $C = X(N,p) := X_1(N) \times_{X(1)} X_0(p)$ where $(N,p)=1$ and $N>4$, $Z = E(N,p)$, the universal generalized elliptic curve over C with level structure and $f: Z \longrightarrow C$ is the natural map, then if one takes P to be the divisor of cusps on C, the quadruple (C, Z, f, P) satisfies the above conditions.

If $h,i,j \geq 0$, $S^{h\,i\,j}(Z/C,P)$ will denote the h-th hypercohomology group of the complex of sheaves, $\mathrm{Sym}^j G^i(Z/C,P) \stackrel{\mathrm{Sym}^j D}{\longrightarrow} \mathrm{Sym}^j G^i(Z/C,P) \otimes \Omega^1_{C_K/K}(\log(P_K))$, where

$$G^i(Z/C,P) = K \otimes_V \mathbb{R}^i g_* \Omega^\bullet_{Z^\times/C^\times} = K \otimes_V H^i_{dR}(Z^\times/C^\times)$$

and D is the Gauss-Manin connection.

The group $S^{h\,i\,j}(Z/C,P)$ naturally has a Hodge filtration which we call $\mathscr{F}^{h\,i\,j,\bullet}(Z/C,P)$. After choosing a branch of the p-adic logarithm on $K^\times$, we will use the rigid geometry of Z/C and p-adic integration to produce a K_0-lattice $S^{h\,i\,j}_{int}(Z/C,P)$ in $S^{h\,i\,j}(Z/C,P)$, a linear operator N^{int}_h on this lattice and make a σ-semi-linear operator Φ^{int}_h on $S^{h\,i\,j}(Z/C,P)$ such that $N^{\mathrm{int}}_h \Phi^{\mathrm{int}}_h = p\Phi^{\mathrm{int}}_h N^{\mathrm{int}}_h$.

A four-tuple $(M, F, N, \mathscr{F}^\bullet)$ where M is a finite dimensional vector space over K_0, F and N are σ-semi-linear and respectively linear operators on M such that $NF = pFN$ and $\mathscr{F}^\bullet$ is a decreasing exhaustive filtration of $M_K := M \otimes_{K_0} K$ by K-vector subspaces is called a filtered, Frobenius, monodromy (FFM) module over K (see **[19]**). The category of FFM-modules is an additive, tensor category with kernels, cokernels and a notion of short exact sequences but it is not abelian. Its subcategory of weakly admissible modules (which are now known to be admissible by [**13**]) is abelian, see also [**19**]. To a $\mathbb{Q}_p$-representation of G_K, Fontaine associated an FFM-module and if this representation "comes from geometry" one can recover it from the FFM-module.

In particular, if $g\colon Z \to C$ is as above then

$$M_{int}^{h\,i\,j}(Z/C,P) := (S_{\rm int}^{h\,i\,j}(Z/C,P), \Phi_h^{\rm int}, N_h^{\rm int}, \mathscr{F}^{h\,i\,j,\bullet}(C,P))$$

is an FFM-module over K.

We will prove,

Theorem 1.1. — *The FFM-module $M_{int}^{h\,i\,j}(Z/C,P)$ is the one associated to*

$$\mathscr{V}^{h\,i\,j}(Z/C,P) := H_{\acute{e}t}^h((C-P)_{\overline{K}}, {\rm Sym}^j(R^i g_{*,\acute{e}t}\mathbb{Q}_p))$$

via Fontaine theory. In particular,

$$\mathscr{V}^{h\,i\,j}(Z/C,P) \cong \big(B_{\rm st}\otimes(M_{\rm int}^{h\,i\,j}(Z/C,P))\big)^{\Phi=Id,N=0}\cap {\rm Fil}^0\big(B_{\rm dR}\otimes_K M_{\rm int}^{h,i,j}(Z/C,P)_K\big).$$

We obtain our theorem from results of Faltings [**17**], which we now describe.

Let us denote by $\overline{C}^{\times}$ the scheme $\overline{C}$ with the inverse image log structure from $C^{\times}$. Suppose $\mathscr{E}$ is a filtered logarithmic F-isocrystal on $\overline{C}^{\times}$. Such an object associates to the "enlargements" (thickenings) of $\overline{C}^{\times}$ (see [**32**] for the non-logarithmic case and [**16**], [**34**],[**35**] in general) coherent sheaves in a compatible way. We will recall the precise definitions in Sections 3.3 and 6. The notion of an F-isocrystal and it's initial development is due to Berthelot and Ogus [**2**], [**32**]. The notion of a filtered logarithmic F-isocrystal was defined by Faltings in [**16**] and developed by Shiho in [**34**] and [**35**]. In particular, one gets from $\mathscr{E}$ a coherent sheaf $\mathscr{E}_{C^{\times}}$ on C_K with an integrable connection D with logarithmic singularities at P. Therefore, if g, Z, C and P are as above, there is a filtered log-F isocrystal $\mathscr{E}_{Z/C}^{i\,j}$ on $\overline{C}^{\times}$ which associates to the enlargement $C^{\times}$, ${\rm Sym}^j G^i(Z/C,P)$.

In [**17**], Faltings associated étale local systems on C, $\mathbb{L}(\mathscr{E})$ to certain (very special) filtered log-F isocrystals, $\mathscr{E}$, and made families of FFM-modules, $(H_{\rm deg}^h(\mathscr{E}), \Phi_h^{\rm deg}, N_h^{\rm deg}, \mathscr{F}_{\rm deg}^{h,\bullet})$ (see Section 2.1 for more details). Let us very briefly describe $H_{\rm deg}^h(\mathscr{E})$. It is the log crystalline cohomology on $\overline{C}$, with a certain log structure $\overline{C}^{\times\times}$, with values in $\mathscr{E}$. As $\overline{C}$ is a reduced divisor with normal crossings in C, let $C^{\times\times}$ be C with the log-structure induced by $\overline{C}\cup P$. Let $\overline{C}^{\times\times}$ be $\overline{C}$ with the pull back log structure. Similarly, let ${\rm Spec}(V)^{\times}$ be ${\rm Spec}(V)$ with the log structure given by the closed point, let ${\rm Spec}(k)^{\times}$ be ${\rm Spec}(k)$ with the pull-back log structure and let ${\rm Spec}(W)^{\times}$ be ${\rm Spec}(W)$ with the Teichmüller lift of the log structure on ${\rm Spec}(k)^{\times}$. Then $\mathscr{E}$ is a filtered log F-isocrystal on $\overline{C}^{\times\times}$ over ${\rm Spec}(W)^{\times}$ and we set $H_{\rm deg}^h(\mathscr{E}) := H_{\rm cris}^h(\overline{C}^{\times\times}/{\rm Spec}(W)^{\times}, \mathscr{E})$ for $h \geq 0$. It is proved in [**17**] that the étale cohomology $H_{\rm et}^h((C-P)_{\overline{K}}, \mathbb{L}(\mathscr{E}))$ and these FFM-modules are associated to each other via Fontaine's theory. In the case, $\mathscr{E} = \mathscr{E}_{Z/C}^{i\,j}$, $H_{\rm deg}^h(\mathscr{E})\otimes_{K_0} K = S^{h\,i\,j}(Z/C,P)$,

$\mathscr{F}_{\deg}^{h,\bullet}$ is the Hodge filtration and $H_{\mathrm{et}}^h((C-P)_{\overline{K}}, \mathbb{L}(\mathscr{E})) = \mathscr{V}^{hij}(Z/C,P)$. In this paper, we will extend the definitions in [C1] of FFM-modules $H_{int}^h(\mathscr{E})$ to regular (see Section 6) logarithmic F-isocrystals $\mathscr{E}$ on $\overline{C}^{\times}$ over Spec(W) and prove

$$H_{\deg}^h(\mathscr{E}) = H_{\mathrm{int}}^h(\mathscr{E})$$

for all $h \geq 0$, when all the irreducible components of $\overline{C}$ are absolutely irreducible.

We have several applications of our theorem. We first point out that our descriptions of the operators Φ_h^{int}, N_h^{int} are more explicit than those of the corresponding operators defined by Hyodo-Kato in ([**23**]) and Faltings in ([**17**]). If $C = X(N,p)$, with $(N,p)=1$ and $N>4$ (see the notations above) and $\mathscr{E} = \mathrm{Sym}^j G^1(E/C,P)$ then we prove that the rank of $N_1^{\deg}$ on $H_{\mathrm{cris}}^1(\overline{C}^{\times,\times}/\mathrm{Spec}(W)^{\times}, \mathscr{E})^{p-new}$ is exactly half the dimension over K_0 of this vector space (see Corollary 7.4.) As a consequence we derive that if f is a p-new cuspidal eigenform of weight $k=j+2$ on $X(N,p)$ and V_f denotes the p-adic G_K-representation attached to f, then V_f is semi-stable but *not* crystalline (Corollary 7.5). This was proved in [**33**] in a very indirect way, using the local Langlands correspondence and results of Carayol on the rank of the monodromy operator on the ℓ-adic ($\ell \neq p$) Weil-Deligne representation attached to f.

Our main result is also used in [**24**] in order to give an explicit description of the image of the p-adic Abel-Jacobi map applied to Heegner cycles on certain Shimura curves in terms of extension classes in the category of FFM-modules. In particular a p-adic Gross-Zagier formula for higher weight modular forms is proved in that paper.

Finally, another application of our results is to get an explicit description of the Mazur-Tate-Teitelbaum $\mathscr{L}$-invariants which we now describe.

Suppose now that $k \geq 0$ is an integer and $(M,F,N,\mathscr{F}^{\bullet})$ is a FFM-module over K such that $\mathscr{F}^iM$ is M_K for $i \leq k$ and it is 0 for $i \geq k+2$. Suppose $\mathscr{H}$ is a commutative $\mathbb{Z}_p$-algebra free of finite rank which acts on M such that $\mathscr{F}^{k+1}M$ is a rank 1 $\mathscr{H}_{\mathbb{Q}_p} := \mathscr{H} \otimes \mathbb{Q}_p$-submodule,

$$M_K = \mathscr{F}^{k+1}M \oplus (N \otimes 1_K)M_K$$

and $N \otimes 1_K \colon \mathscr{F}^{k+1}M \longrightarrow (N\otimes 1_K)M_K$ is a non-zero $\mathscr{H}_{\mathbb{Q}_p}$-isomorphism. Then, if $v \in M$ is an eigenvector for F such that $(N \otimes 1_K)M_K = \mathscr{H}_{\mathbb{Q}_p} \cdot Nv$, the $\mathscr{L}$-invariant $\mathscr{L}(M)$ of $(M,F,N,\mathscr{F}(D)^{\bullet})$ is the unique element in $\mathscr{H}_{\mathbb{Q}_p}$ such that

$$v - \mathscr{L}(M)Nv \in \mathscr{F}^{k+1}M.$$

The general definition of an $\mathscr{L}$-invariant becomes arithmetically significant when we attach it to a cuspidal newform on $X(N,p)$ of weight $k+2$ (as above), with $k \geq 0$ even, which is split multiplicative at p. This means precisely that $a_p = p^{k/2}$ (see [**29**].) The quest for an $\mathscr{L}$-invariant which is intimately connected to the relationship between complex and p-adic L-functions was initiated by Mazur-Tate-Teitelbaum (86) in [**30**]. There, a definition in the weight 2 case was offered. Its relationship with values of L-functions was established by Greenberg and Stevens using Hida theory (91) in [**20**]. Teitelbaum proposed the first definition in the higher weight case under some restrictions on the level using the uniformization of Shimura curves by the p-adic upper half plane (90) in [**38**] (his definition does not involve a FFM-module but see [**24**]), the first author of the present paper offered a definition using the FFM-module $M_{int}^{1\,i\,j}(E(N,p)/X(N,p), \mathrm{Cusps})$ and $\mathscr{H}$ is the Hecke-algebra acting on $X(N,p)$, in [**8**]. Finally, Fontaine-Mazur defined an $\mathscr{L}$-invariant associated to a cusp form as above using the FFM-module $D_{st}(V)$, where V is the local Galois representation attached to the cusp form and D_{st} is Fontaine's functor (see [**19**]) in [**29**]. The algebra $\mathscr{H}$ is again the Hecke algebra acting on $X(N,p)$. K. Kato, M. Kurihara and T. Tsuji established the connection between the $\mathscr{L}$-invariant of Fontaine and Mazur and special values of the complex and p-adic L-functions while G. Stevens has established the connection between the $\mathscr{L}$-invariant defined in [**8**] and special values of the complex and p-adic L-functions using p-adic families of modular forms, see [**37**]. The result of Kato, Kurihara and Tsuji has not yet been published. The present paper together with the results in [**24**] establishes the equality of all the $\mathscr{L}$-invariants (whenever they are defined). Of course, the results of Kato-Kurihara-Tsuji and Stevens togeher also imply (indirectly) the equality of the $\mathscr{L}$-invariants defined in [**8**] and the corresponding Fontaine-Mazur $\mathscr{L}$-invariants.

We mention that P. Colmez also proved (in [**12**]) a formula giving the $\mathscr{L}$-invariant of Fontaine-Mazur as derivative of a family of eigenvalues of Frobenius. Together with the result of Stevens mentioned above involving the $\mathscr{L}$-invariant defined in [**8**], this gives another local proof of the equality of the two $\mathscr{L}$-invariants we consider.

In [**21**] Grosse-Klönne extended the Hyodo-Kato theory and showed that there are natural Frobenius and monodromy operators on the de Rham cohomology of a quite general rigid space. He has been able to explicitly compute these when the space is a quotient of a p-adic symmetric domain.

Writing this paper we had two options, namely to present the definitions, statements and proofs in the most general case (the logarithmic case), which would have made the notations very complicated and would have obscured the ideas of the proofs or, to first present some of the definitions, statements and proofs in the non-logarithmic case, then to give the definitions and make the precise statements in general and leave it to the reader to check that the same proofs go through with the obvious adjustments. We choose to do the latter.

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Thus, it should be understood that the paper owes much to this report and we are very grateful to its author for his/her help.

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