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HOMOGENEOUS COMMUTING VECTOR FIELDS ON $\mathbb{C}^2$

by

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Abstract. — In the main result of this paper we give a method to construct all pairs of homogeneous commuting vector fields on $\mathbb{C}^2$ of the same degree $d \geq 2$ (Theorem 1). As an application, we classify, up to linear transformations of $\mathbb{C}^2$, all pairs of commuting homogeneous vector fields on $\mathbb{C}^2$, when $d = 2$ and $d = 3$ (corollaries 1 and 2). We obtain also necessary conditions in the cases of quasi-homogeneous vector fields and when the degrees are different (theorem 2).

***Résumé* (Champs de vecteurs homogènes commutants dans $\mathbb{C}^2$).** — Dans le résultat principal de ce papier on donne une méthode de construction de tous les paires de champs de vecteurs homogènes de même degré $d \geq 2$ qui commutent (théorème 1). Comme application, on classifie les paires de champs de vecteurs homogènes commutantes dans $\mathbb{C}^2$ de degrés $d = 2$ et $d = 3$ (corollaires 1 et 2). Nous obtenons aussi des conditions nécessaires dans les cas quasi-homogènes et quand les degrés sont différents (théorème 2).

1. Introduction

A. Guillot in his thesis and in [**3**], gave a non-trivial example of a pair of commuting homogeneous vector fields of degree two on $\mathbb{C}^3$. The example is non-trivial in the sense that it cannot be reduced to two vector fields in separated variables, like in the pair $X := P(x,y)\partial_x + Q(x,y)\partial_y$ and $Y := R(z)\partial_z$. This suggested me the problem of classification of pairs of polynomial commuting vector fields on $\mathbb{C}^n$. This problem, in this generality, seems very difficult, even for $n = 2$. Even the restricted problem of classification of pairs of commuting vector fields, homogeneous of degree d, seems very difficult for $n \geq 3$ and $d \geq 2$ (see problem 3). However, for $n = 2$ and $d \geq 2$ it is possible to give a complete classification, as we will see in this paper.

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Let X and Y be two homogeneous commuting vector fields on $\mathbb{C}^2$, where $dg(X) = k$ and $dg(Y) = \ell$, and $R = x\,\partial_x + y\,\partial_y$ be the radial vector field.

Definition 1.1. — *We will say that X and Y are* colinear *if $X \wedge Y = 0$. In this case, we will use the notation $X//Y$. When $dg(X) = dg(Y)$, we will consider the 1-parameter family $(Z_\lambda)_{\lambda\in\mathbb{P}^1}$ given by $Z_\lambda = X + \lambda.Y$ if $\lambda \in \mathbb{C}$ and $Z_\infty = Y$. It will be called the* pencil generated *by X and Y. The pencil will be called* trivial, *if $Y = \lambda.X$ for some $\lambda \in \mathbb{C}$. Otherwise, it will be called* non-trivial.

From now on, we will set:

$$\begin{cases} X \wedge Y = f\,\partial_x \wedge \partial_y \\ R \wedge X = g\,\partial_x \wedge \partial_y \\ R \wedge Y = h\,\partial_x \wedge \partial_y \end{cases} . \tag{1}$$

Since $dg(X) = k$ and $dg(Y) = \ell$, the polynomials f, g and h are homogeneous and $dg(f) = k + \ell$, $dg(g) = k + 1$, $dg(h) = \ell + 1$. Moreover, $f \not\equiv 0$ iff X and Y are non-colinear.

Our main result concerns the case where $k = \ell \geq 2$. In this case, if $g, h \not\equiv 0$, we will consider the meromorphic function $\phi = g/h$ as a holomorphic function $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$:

$$\phi[x : y] = \frac{g(x,y)}{h(x,y)} .$$

Theorem 1. — *Let $(Z_\lambda)_\lambda$ be a non-trivial pencil of homogeneous commuting vector fields of degree $d \geq 2$ on $\mathbb{C}^2$. Let X and Y be two generators of the pencil and f, g, h and ϕ be as before. If the pencil is colinear then $X = \alpha.R$ and $Y = \beta.R$, where α and β are homogeneous polynomials of degree $d - 1$. If the pencil is non-colinear then:*

(a) *$f, g, h \not\equiv 0$.*

(b) *f/g (resp. f/h) is a non-constant meromorphic first integral of X (resp. Y).*

(c) *Let s be the (topological) degree of $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$. Then $1 \leq s \leq d - 1$.*

(d) *The decompositions of f, g and h into irreducible linear factors are of the form:*

$$\begin{cases} f = \Pi_{j=1}^{r} f_j^{2k_j+m_j} \\ g = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_i \\ h = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} h_i \end{cases} \tag{2}$$

where $s + \sum_{j=1}^{r} k_j = d + 1$ and $\sum_{j=1}^{r} m_j = 2s - 2$. Moreover, we can choose the generators X and Y in such a way that $g_1, \ldots, g_s, h_1, \ldots, h_s$ are two by two relatively prime.

(e) *Considering the direction $(f_j = 0) \subset \mathbb{C}^2$ as a point $p_j \in \mathbf{P}^1$, then*

$$m_j = \operatorname{mult}(\phi, p_j) - 1 \ , \ j = 1, \ldots, r \ , \tag{3}$$

where $\operatorname{mult}(\phi, p)$ denotes the ramification index of ϕ at $p \in \mathbf{P}^1$.

(f) *The generators X and Y can be chosen as:*

$$(4)\qquad \begin{cases} X = g.[\sum_{j=1}^{r}(k_j+m_j)\frac{1}{f_j}(f_{jx}\partial_y - f_{jy}\partial_x) - \sum_{i=1}^{s}\frac{1}{g_i}(g_{ix}\partial_y - g_{iy}\partial_x)] \\ Y = h.[\sum_{j=1}^{r}(k_j+m_j)\frac{1}{f_j}(f_{jx}\partial_y - f_{jy}\partial_x) - \sum_{i=1}^{t}\frac{1}{h_i}(h_{ix}\partial_y - h_{iy}\partial_x)] \end{cases}$$

Conversely, given a non-constant map $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ of degree $s \geq 1$ and a divisor D on $\mathbf{P}^1$ of the form

$$(5)\qquad D = \sum_{p\in\mathbf{P}^1} (2k(p) + \mathrm{mult}(\phi,p) - 1).[p]\ ,$$

where $k(p) \geq min(1, \mathrm{mult}(\phi,p)-1)$ and $\sum_p k(p) < +\infty$, there exists a unique pencil $(Z_\lambda)_\lambda$ of homogeneous commuting vector fields of degree $d = \sum_p k(p) + s - 1$ with generators X and Y given by (4), and the f_j's, g_i's and h_i's given in the following way: let $\{p_1 = [a_1 : b_1], \ldots, p_r = [a_r : b_r]\} = \{p \in \mathbf{P}^1 \mid 2k(p) + \mathrm{mult}(\phi,p) - 1 > 0\}$. Set $k_j = k(p_j)$, $m_j = \mathrm{mult}(\phi,p_j) - 1$ and $f_j(x,y) = a_j\, y - b_j\, x$. Set $\phi[x : y] = G_1(x,y)/H_1(x,y)$, where G_1 and H_1 are homogeneous polynomials of degree s. Then the g_i's and h_i's are the linear factors of G_1 and H_1, respectively.

Definition 1.2. — *Let X, Y, $g = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_i$ and $h = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} h_i$ be as in theorem 1. We call $(f_j = 0)$, $j = 1, \ldots, r$, the fixed directions of the pencil.*

Given $\lambda \in \mathbb{C}$, the polynomial $g_\lambda = g + \lambda.h$ plays the same role for the vector field $Z_\lambda = X + \lambda.Y$ as g and h for X and Y. Its decomposition into irreducible factors is of the form

$$g_\lambda = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_{i,\lambda}\ .$$

Definition 1.3. — *The directions given by $(g_{i,\lambda} = 0)$ are called the movable directions of the pencil.*

In particular, the number s of movable directions coincides with the degree of the map $\phi = g/h\colon \mathbf{P}^1 \to \mathbf{P}^1$.

As an application of Theorem 1, we obtain the classification of the pencils of homogeneous commuting vector fields of degrees two and three.

Corollary 1. — *Let $(Z_\lambda)_\lambda$ be a pencil of commuting homogeneous of degree two vector fields on $\mathbb{C}^2$. Then, after a linear change of variables on $\mathbb{C}^2$, the generators X and Y of the pencil can be written as:*

(a) *$X = g.R$ and $Y = h.R$, where g and h are homogeneous polynomials of degree one and $R = x.\partial_x + y.\partial_y$.*
(b) *$X = x^2\partial_x$ and $Y = y^2\partial_y$. In this case, the pencil has two fixed directions.*
(c) *$X = y^2\partial_x$ and $Y = 2xy\partial_x + y^2\partial_y$. In this case, the pencil has one fixed direction.*

Corollary 2. — *Let $(Z_\lambda)_\lambda$ be a pencil of commuting homogeneous of degree three vector fields on $\mathbb{C}^2$. Then, after a linear change of variables on $\mathbb{C}^2$, the generators X and Y of the pencil can be written as:*

(a) *$X = g.R$ and $Y = h.R$, where g and h are homogeneous polynomials of degree two and $R = x.\partial_x + y.\partial_y$.*
(b) *$X = y^3\partial_x$ and $Y = 3xy^2\partial_x + y^3\partial_y$. In this case, the pencil has one movable and one fixed direction.*
(c) *$X = x^2y\partial_x$ and $Y = xy^2\partial_x - y^3\partial_y$. In this case, the pencil has one movable and two fixed directions.*
(d) *$X = (2x^2y + x^3)\partial_x - x^2y\partial_y$ and $Y = -xy^2\partial_x + (2xy^2 + y^3)\partial_y$. In this case, the pencil has one movable and three fixed directions.*
(e) *$X = x^3\partial_x$ and $Y = y^3\partial_y$. In this case, the pencil has two movable and two fixed directions.*

Some of the preliminary results that we will use in the proof of Theorem 1 are also valid for quasi-homogeneous vector fields.

Definition 1.4. — *Let S be a linear diagonalizable vector field on $\mathbb{C}^n$ such that all eigenvalues of S are relatively prime natural numbers. We say that a holomorphic vector field $X \not\equiv 0$ is quasi-homogeneous with respect to S if $[S, X] = m\,X$, $m \in \mathbb{C}$.*

It is not difficult to prove that, in this case, we have the following:
(I) $m \in \mathbb{N} \cup \{0\}$.
(II) X is a polynomial vector field.

Our next result concerns two commuting vector fields which are quasi-homogeneous with respect to the same linear vector field S. Let X and Y be two commuting vector fields on $\mathbb{C}^2$, quasi-homogeneous with respect to the same vector field S with eigenvalues $p, q \in \mathbb{N}$ (relatively prime), where $[S, X] = m\,X$ and $[S, Y] = n\,Y$. Since S is diagonalizable, after a linear change of variables, we can assume that $S = p\,x\partial_x + q\,y\partial_y$. Set $X \wedge Y = f\,\partial_x \wedge \partial_y$, $S \wedge X = g\,\partial_x \wedge \partial_y$ and $S \wedge Y = h\,\partial_x \wedge \partial_y$. We will always assume that $X, Y \not\equiv 0$

Remark 1.1. — *We would like to observe that f, g and h are quasi-homogeneous with respect to S, that is, we have $S(f) = (m + n + tr(S))f$, $S(g) = (m + tr(S))g$ and $S(h) = (n + tr(S))h$, where $tr(S) = p + q$. It is known that in this case, any irreducible factor of f, g or h, is the equation of an orbit of S, that is, x, y or a polynomial of the form $y^p - c\,x^q$, where $c \neq 0$.*

Theorem 2. — *In the above situation, suppose that $f, h \not\equiv 0$ and $n \neq 0$. Then:*
(a) *$g \not\equiv 0$ and f/g is a non-constant meromorphic first integral of X.*
(b) *Suppose that $m, n \neq 0$. Then f, g and h satisfy the two equivalent relations below:*

$$mn\,f^2\,dx \wedge dy = f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg \tag{6}$$

$$(m - n)\frac{df}{f} + n\frac{dh}{h} - m\frac{dg}{g} = \frac{m\,n\,f}{gh}(qy\,dx - px\,dy) \tag{7}$$

(c) *Suppose that $m, n \neq 0$. Then any irreducible factor of f divides g and h. Conversely, if $p = gcd(g,h)$ then any irreducible factor of p divides f. Moreover, the decompositions of f, g and h into irreducible factors, are of the form*

$$\begin{cases} f = \Pi_{j=1}^{r} f_j^{\ell_j} \\ g = \Pi_{j=1}^{r} f_j^{m_j} . \Pi_{i=1}^{s} g_i^{a_i} \\ h = \Pi_{j=1}^{r} f_j^{n_j} . \Pi_{i=1}^{t} h_i^{b_i} \end{cases} \tag{8}$$

where $r > 0$, $m_j, n_j > 0$, $\ell_j \geq m_j + n_j - 1$, for all j, and any two polynomials in the set $\{f_1, \dots, f_r, g_1, \dots, g_s, h_1, \dots, h_t\}$ are relatively prime.

(d) *Suppose that f, g and h are as in (8). Then vector fields X and Y can be written as*

$$\begin{cases} X = \frac{1}{n} g . [\sum_{j=1}^{r} (\ell_j - m_j) \frac{1}{f_j} (f_{jx} \partial_y - f_{jy} \partial_x) - \sum_{i=1}^{s} a_i \frac{1}{g_i} (g_{ix} \partial_y - g_{iy} \partial_x)] \\ Y = \frac{1}{m} h . [\sum_{j=1}^{r} (\ell_j - n_j) \frac{1}{f_j} (f_{jx} \partial_y - f_{jy} \partial_x) - \sum_{i=1}^{t} b_i \frac{1}{h_i} (h_{ix} \partial_y - h_{iy} \partial_x)] \end{cases} \tag{9}$$

As an application, we have the following result:

Corollary 3. — *Let X and Y be germs of holomorphic commuting vector fields at $0 \in \mathbb{C}^2$. Let*

$$X = \sum_{j=d}^{\infty} X_j$$

be the Taylor series of X at $0 \in \mathbb{C}^2$, where X_j is homogeneous of degree $j \geq d$. Assume that $d \geq 2$ and that the vector field X_d has no meromorphic first integral and that 0 is an isolated singularity of X_d. Then $Y = \lambda . X$, where $\lambda \in \mathbb{C}$.

We would like to recall a well-known criterion for a homogeneous vector field of degree d on $\mathbb{C}^2$, say X_d, to have a meromorphic first integral (see [**1**]). Since the radial vector field $R = x\,\partial_x + y\,\partial_y$ has the meromorphic first integral y/x, we can assume that $R \wedge X_d = g\,\partial_x \wedge \partial_y \not\equiv 0$. Let $\omega = i_{X_d}(dx \wedge dy)$, where i denotes the interior product. Then the form $\omega_1 = \omega/g$ is closed. In this case, if $g = \Pi_{j=1}^{r} g_j^{k_j}$ is the decomposition of g into linear irreducible factors, then we have

$$\omega_1 = \sum_{j=1}^{r} \lambda_j \frac{dg_j}{g_j} + d(h/g_1^{k_1-1} \cdots g_r^{k_r-1}) ,$$

where $\lambda_j \in \mathbb{C}$, for all $1 \leq j \leq r$ and h is homogeneous of degree $d + 1 - r = dg(X_d) + 1 - r = dg(g/g_1 \cdots g_r)$. In this case, X_d has a meromorphic first integral if, and only if, either $\lambda_1 = \cdots = \lambda_r = 0$, or $\lambda_j \neq 0$ for some $j \in \{1, \dots, r\}$, $h \equiv 0$ and $[\lambda_1 : \cdots : \lambda_r] = [m_1 : \cdots : m_r]$, where $m_1, \dots, m_r \in \mathbb{Z}$. In particular, we obtain that the set of homogeneous vector fields of degree $d \geq 1$ with a meromorphic first integral is a countable union of Zariski closed sets.

Let us state some natural problems related to the above results.

Problem 1. — *Classify the pencils of commuting homogeneous vector fields of degree $d \geq 2$ on $\mathbb{C}^n$, $n \geq 3$.*