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PROJECTIVE STRUCTURES AND PROJECTIVE BUNDLES OVER COMPACT RIEMANN SURFACES

by

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To José Manuel Aroca for his 60[th] birthday

Abstract. — A projective structure on a compact Riemann surface C of genus g is given by an atlas with transition functions in $\mathrm{PGL}(2,\mathbb{C})$. Equivalently, a projective structure is given by a $\mathbb{P}^1$-bundle over C equipped with a section σ and a foliation $\mathscr{F}$ which is both transversal to the $\mathbb{P}^1$-fibers and the section σ. From this latter geometric bundle picture, we survey on classical problems and results on projective structures. By the way, we will recall some basic facts about $\mathbb{P}^1$-bundles. We will give a complete description of projective (actually affine) structures on the torus with an explicit versal family of foliated bundle picture.

Résumé **(Structures projectives et fibrés projectifs sur les surfaces de Riemann compactes)**
Une structure projective sur une surface de Riemann C de genre g est donnée par un atlas dont les applications de transition sont à valeurs dans $\mathrm{PGL}(2,\mathbb{C})$. De manière équivalente, une structure projective est donnée par un fibré en $\mathbb{P}^1$ sur C équipé d'une section σ et d'un feuilletage $\mathscr{F}$ transverse à la fois aux fibres $\mathbb{P}^1$ et à la section σ. À partir de cette dernière description géométrique, nous survolons quelques problèmes et résultats classiques sur les structures projectives. Nous rappelons quelques propriétés de base sur les fibrés en $\mathbb{P}^1$. Nous donnons une description complète des structures projectives (qui sont en fait affines) sur le tore avec une famille verselle explicite de fibrés feuilletés.

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1. Projective structures

1.1. Definition and examples. — Denote by Σ_g the orientable compact real surface of genus g. A *projective structure* on Σ_g is given by an atlas $\{(U_i, f_i)\}$ of coordinate charts (local homeomorphisms) $f_i : U_i \to \mathbb{P}^1$ such that the transition functions $f_i = \varphi_{ij} \circ f_j$ are restrictions of Moebius transformations $\varphi_{ij} \in \mathrm{PGL}(2, \mathbb{C})$.

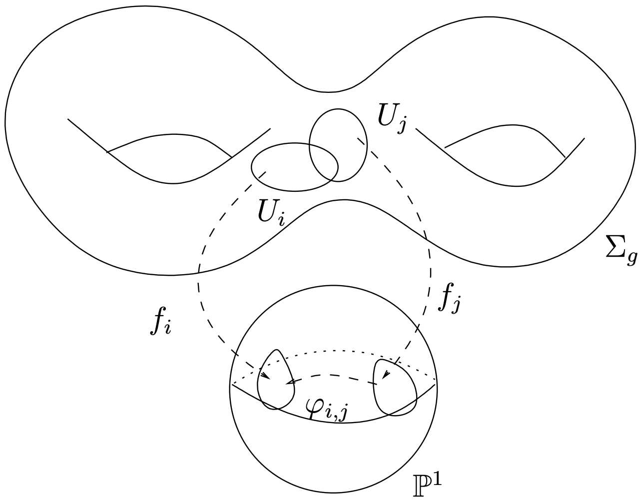

Figure 1. Projective atlas

There is a unique maximal atlas defining the projective structure above, obtained from the previous one by adding all charts $\{(U_i, \varphi \circ f_i)\}$ when φ runs over $\mathrm{PGL}(2, \mathbb{C})$.

A projective structure induces a *complex structure* on Σ_g, just by pulling-back that of $\mathbb{P}^1$ by the projective charts. We will denote by C the corresponding Riemann surface (complex curve).

***Example 1.1* (The Universal cover).** — Let C be a compact Riemann surface having genus g and consider its universal cover $\pi : U \to C$. By the Riemann Mapping Theorem, we can assume that U is either the Riemann sphere $\mathbb{P}^1$, or the complex plane $\mathbb{C}$ or the unit disk Δ depending wether $g = 0$, 1 or ≥ 2. We inherit a representation of the fundamental group $\rho : \pi_1(C) \to \mathrm{Aut}(U)$ whose image Λ is actually a subgroup of $\mathrm{PGL}(2, \mathbb{C})$. All along the paper, by abuse of notation, we will identify elements $\gamma \in \pi_1(C)$ with their image $\rho(\gamma) \in \mathrm{PGL}(2, \mathbb{C})$. The atlas defined on C by all local determinations of $\pi^{-1} : C \dashrightarrow \mathbb{P}^1$ defines a projective structure on C compatible with the complex one. Indeed, any two determinations of π^{-1} differ by left composition with an element of Λ.

We thus see that any complex structure on Σ_g is subjacent to a projective one. In fact, for $g \geq 1$, we will see that there are many projective structures compatible to a given complex one (see Theorem 1.2). We will refer to the projective structure above as the *canonical projective structure* of the Riemann surface C: it does not depend on the choice of the uniformization of U. We now give other examples.

***Example 1.2* (Quotients by Kleinian groups).** — Let $\Lambda \subset \mathrm{PGL}(2,\mathbb{C})$ be a subgroup acting properly, freely and discontinuously on some connected open subset $U \subset \mathbb{P}^1$. Then, the quotient map $\pi : U \to C := U/\Lambda$ induces a projective structure on the quotient C, likely as in Example 1.1. There are many such examples where U is neither a disk, nor the plane. For instance, *quasi-Fuchsian groups* are obtained as image of small perturbations of the representation ρ of Example 1.1; following [**35**], such perturbations keep acting discontinuously on some *quasi-disk* (a topological disk whose boundary is a Jordan curve in $\mathbb{P}^1$).

***Example 1.3* (Schottky groups).** — Pick $2g$ disjoint discs $\Delta_1^-,\dots,\Delta_g^-$ and $\Delta_1^+,\dots,\Delta_g^+$ in $\mathbb{P}^1$, $g \geq 1$. For $i = 1,\dots,n$, let $\varphi_i \in \mathrm{PGL}(2,\mathbb{C})$ be a loxodromic map sending the disc Δ_i^- onto the complement $\mathbb{P}^1 - \Delta_i^+$.

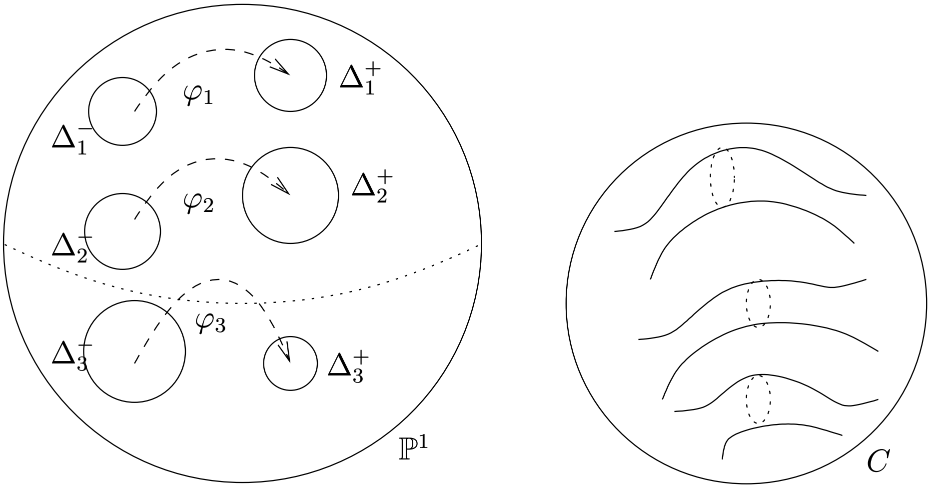

FIGURE 2. Schottky groups

The group $\Lambda \subset \mathrm{PGL}(2,\mathbb{C})$ generated by $\varphi_1,\dots,\varphi_g$ acts properly, freely, and discontinuously on the complement U of some closed set contained inside the disks (a Cantor set whenever $g \geq 2$). The fundamental domain of this action on U is given by the complement of the disks and the quotient $C = U/\Lambda$ is obtained by gluing together the boundaries of Δ_i^+ and Δ_i^- by means of φ_i, $i = 1,\dots,g$. Therefore, C is a compact Riemann surface of genus g. This picture is clearly stable under small deformation of the generators φ_i and we thus obtain a complex $3g-3$ dimensional family of projective structures on the genus g surface Σ_g (we have divided here by the action of $\mathrm{PGL}(2,\mathbb{C})$ by conjugacy).

1.2. Developping map and monodromy representation. — Given a projective atlas and starting from any initial coordinate chart (U_0, f_0), one can extend it analytically along any path γ starting from $p_0 \in U_0$.

Indeed, after covering γ by finitely many projective coordinate charts, say (U_0, f_0), (U_1, f_1), ... ,(U_n, f_n), one can modify them step by step in the following way. First of all, since $f_0 = \varphi_{01} \circ f_1$ on $U_0 \cap U_1$, one can replace the chart f_1 by $\tilde{f}_1 := \varphi_{01} \circ f_1$ which is well-defined on U_1, extending f_0. Next, we replace f_2 by $\tilde{f}_2 := \varphi_{01} \circ \varphi_{12} \circ f_2$ which, on $U_1 \cap U_2$, coincide with $\tilde{f}_1$. Step by step, we finally arrive at the chart $\tilde{f}_n := \varphi_{01} \circ \cdots \circ \varphi_{n-1\ n} \circ f_n$ which, by construction, is the analytic continuation of f_0 along γ.

Therefore, the local chart (U_0, f_0) extends (after lifting on the universal covering) as a global submersion on the universal cover

$$f : U \to \mathbb{P}^1$$

which is called the *developping map* of the projective structure. The developping map is moreover holomorphic with respect to the complex structure subjacent to the projective one. By construction, the monodromy of f along loops takes the form

$$f(\gamma.u) = \varphi_\gamma \circ f, \quad \varphi_\gamma \in \mathrm{PGL}(2, \mathbb{C}) \quad \forall \gamma \in \pi_1(\Sigma_g, p_0) \tag{1}$$

(u is the coordinate on U and $\gamma.u$, the canonical action of $\pi_1(\Sigma_g, p_0)$ on U). In fact, φ_γ is the composition of all transition maps $\varphi_{i,j}$ encoutered along γ for a given finite covering of projective charts: with notations above, setting $(U_n, f_n) = (U_0, f_0)$, we have $\varphi_\gamma = \varphi_{01} \circ \cdots \circ \varphi_{n-1\ n}$. It turns out that φ_γ only depends on the homotopy class of γ and we inherit a *monodromy representation*

$$\rho : \pi_1(\Sigma_g, p_0) \to \mathrm{PGL}(2, \mathbb{C}) \ ; \ \gamma \mapsto \varphi_\gamma. \tag{2}$$

The image Λ of ρ will be called monodromy group. The developping map f is defined by the projective structure up to the choice of the initial chart (U_0, f_0) above: it is unique up to left composition $\varphi \circ f$, $\varphi \in \mathrm{PGL}(2, \mathbb{C})$. Therefore, the monodromy representation is defined by the projective structure up to conjugacy: the monodromy of $\varphi \circ f$ is $\gamma \mapsto \varphi \circ \varphi_\gamma \circ \varphi^{-1}$.

Conversely, any global submersion $f : U \to \mathbb{P}^1$ on the universal covering $\pi : U \to \Sigma_g$ satisfying (1) is the developping map of a unique projective structure on Σ_g. We note that condition (1) forces the map $\gamma \to \varphi_\gamma$ to be a morphism.

Example 1.4. — The developping map of the canonical projective structure (see example 1.1) is the inclusion map $U \hookrightarrow \mathbb{P}^1$ of the universal cover of C. More generally, when the projective structure is induced by a quotient map $\pi : U \to C = U/\Lambda$ like in example 1.2, then the developping map f is the universal cover $\tilde{U} \to U$ of U and the monodromy group is Λ. In example 1.3, the open set U is not simply connected (the complement of a Cantor set) and the developping map is a non trivial covering. Thus the corresponding projective structure is not the canonical one. Similarly, the developping map of a quasi-Fuchsian group is the uniformization map of the corresponding quasi-disk and is not trivial; the projective structure is neither the canonical one, nor of Schottky type.

***Example 1.5* (The Sphere).** — Given a projective structure of the Riemann sphere $\mathbb{P}^1$, we see that the developping map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is uniform (no monodromy since $\pi_1(\mathbb{P}^1)$ is trivial). Therefore, f is a global holomorphic submersion (once we have fixed the complex structure) and thus $f \in \mathrm{PGL}(2, \mathbb{C})$. Consequently, the projective structure is the canonical one and it is the unique projective structure on $\mathbb{P}^1$.

For similar reason, we remark that the monodromy group of a projective structure on a surface of genus $g \geq 1$ is never trivial.

***Example 1.6* (The Torus).** — Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice in $\mathbb{C}$, $\tau \in \mathbb{H}$, and consider the elliptic curve $C := \mathbb{C}/\Lambda$. The monodromy of a projective structure on C is abelian; therefore, after conjugacy, it is in one of the following abelian groups:

- the linear group $\{\varphi(z) = az \ ; \ a \in \mathbb{C}^*\}$,
- the translation group $\{\varphi(z) = z + b \ ; \ b \in \mathbb{C}\}$,
- the finite abelian dihedral group generated by $-z$ and $1/z$.

The canonical projective structure on C has translation monodromy group Λ. On the other hand, for any $c \in \mathbb{C}^*$ the map

$$f_c : \mathbb{C} \to \mathbb{P}^1 \ ; \ u \mapsto \exp(c.u) \tag{3}$$

is the developping map of a projective structure on C whose monodromy is linear, given by

$$f_c(u+1) = e^c \cdot f(u) \quad \text{and} \quad f_c(u+\tau) = e^{c\tau} \cdot f(u). \tag{4}$$

We inherit a 1-parameter family of projective structures parametrized by $c \in \mathbb{C}^*$ (note that $f_0 \equiv 1$ is not a submersion). We will see latter that this list is exhaustive. In particular, all projective structures on the torus are actually affine (transition maps in the affine group).

The projective structures listed in example 1.6 are actually *affine structures*: the developping map takes values in $\mathbb{C}$ with affine monodromy.

***Theorem 1.1* (Gunning [12]).** — *All projective structures on the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, are actually affine and listed in example 1.6 above. There is no projective structure having affine monodromy on surfaces Σ_g of genus $g \geq 2$.*

In particular, the dihedral group is not the holonomy group of a projective structure on the torus.

Partial proof. — Here, we only prove that the list of example 1.6 exhausts all affine structures on compact Riemann surfaces. In example 1.7, we will see that there are no other projective structure on tori than the affine ones.