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ON GENERALIZED SURFACES IN $(\mathbb{C}^3, 0)$

by

Percy Fernández-Sánchez & Jorge Mozo-Fernández

Dedicated to J.M. Aroca for his sixtieth birthday

En esto fueron razonando los dos, hasta que llegaron a un pueblo
donde fue ventura hallar un algebrista, con quien se curó el Sansón desgraciado.
El Ingenioso Hidalgo Don Quijote de la Mancha

Abstract. — In this paper we study germs of codimension one holomorphic, non-dicritical, singular foliations in $(\mathbb{C}^3, 0)$ having no saddle-nodes in their reduction of singularities. These are called *generalized surfaces*. The main result says that the reduction of the singularities of a generalized surface agrees with the reduction if its separatrix set.

Résumé **(Sur les surfaces généralisées dans** $(\mathbb{C}^3, 0)$**).** — Dans cet article, on étudie les germes de feuilletages holomorphes de codimension un, non dicritiques et singuliers en $(\mathbb{C}^3, 0)$, qui n'ont pas de selles-nœuds dans la réduction des leurs singularités. Ces feuilletages s'appellent *surfaces généralisées*. Le résultat principal affirme que la réduction des singularités d'une surface généralisée coïncide avec la réduction de son ensemble de séparatrices.

1. Introduction

The main objective of this paper is to characterize a class of holomorphic codimension one foliations in $(\mathbb{C}^3, 0)$, whose reduction of singularities agrees with the reduction of their separatrices. We will call *generalized surfaces* to these foliations, as they are a generalization of the notion of generalized curves introduced by C. Camacho, A. Lins

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Neto and P. Sad in [**2**]. Following that paper, a codimension one singular foliation $\mathcal{F}$ defined by a 1-form ω in $(\mathbb{C}^2, 0)$ will be called a generalized curve if:

1. $\mathcal{F}$ is not dicritical, i.e., it has a finite number of separatrices, or equivalently the exceptional divisor obtained after reduction of singularities is invariant by $\mathcal{F}$.
2. There are no saddle-nodes in the reduction of the singularities of $\mathcal{F}$.

Recall that a germ of singular foliation $\mathcal{F}$ defined by ω is called reduced, or simple, if there are coordinates (x, y) such that

$$\omega = (\lambda x + h.o.t.)dy + (\mu y + h.o.t.), \quad \mu \neq 0, \text{ and } \frac{\lambda}{\mu} \notin \mathbb{Q}_{<0}.$$

If $\lambda = 0$, the singularity is called a saddle-node.

Generalized curves have a number of good properties, as shown in [**2**]. If $(f = 0)$ is an analytic equation of their set S of separatrices, then:

1. $\nu(\omega) = \nu(df)$, where ν denotes the order at 0.
2. The reduction of the singularities of $\mathcal{F}$ is the same as the reduction of $(f = 0)$.
3. $\mu(\omega) = \mu(f)$, where μ denotes the Milnor number of ω, f, at the origin. Note that for every foliation we always have the inequality $\mu(\omega) \geq \mu(f)$, and the equality only holds for generalized curves.

Another properties of generalized curves have been established more recently. For instance, let $BB(\mathcal{F}, P)$ denote the Baum-Bott index of $\mathcal{F}$ at P, $CS(\mathcal{F}, S, P)$ the Camacho-Sad index of $\mathcal{F}$ at P relative to the invariant curve S, and $GSV(\mathcal{F}, S, P)$ the index of Gómez-Mont, Seade and Verjovsky. Then, M. Brunella shows in [**1**] that, for a generalized curve, $GSV(\mathcal{F}, S, P) = 0$, and that this implies $BB(\mathcal{F}, P) = CS(\mathcal{F}, S, P)$.

In our previous paper [**12**], we classified analytically a class of singular foliations in $(\mathbb{C}^3, 0)$, which we called quasi-ordinary cuspidal foliations. These are foliations with one separatrix, that is a cuspidal, quasi-ordinary surface. In particular it can be seen in that paper that, after resolving the singularities of the surface, the singularities of the foliation are automatically reduced, and the analytic classification of these foliations agrees with the analytic classification of the projective holonomy of a certain component of the exceptional divisor. This property is similar to the property of generalized curves we saw before, and it motivates the generalization of this notion to dimension three. In fact, in dimension three it exists a theorem of reduction of singularities for holomorphic foliations [**6, 5**], and a theorem of existence of separatrices (in the non-dicritical case) [**6**] that allows the generalization.

The plan of the paper is as follows: in section 2, we shall recall some basic facts about holomorphic foliations, mainly about the final forms (simple singularities). In section three, we will give the definition of generalized surface, and will show its main properties. The main result of this paper will then be as follows.

Theorem 1.1. — *If $\mathcal{F}$ is a generalized surface in $(\mathbb{C}^3, 0)$, and $S \equiv (f = 0)$ is the union of their separatrices, then the reduction of singularities of $\mathcal{F}$ agrees with the reduction of the surface S.*

Of course, from a careful study of [**6**] the main result of this paper can be deduced, but it is not explicit. The purpose of this paper is to offer an independent proof more in the spirit of [**2**]. It is also worth to mention here related works about foliations in dimension three that admit particular reduction of singularities, for instance foliations that can be desingularized with only punctual blow-ups [**7**].

There is work in progress about the analytic classification of generalized curves. For instance, several papers have been written about the classification of quasi-homogeneous singularities of foliations (see [**17**] for definitions and statements). Let us mention the works of D. Marín [**15, 16**], and more recently, Y. Genzmer [**13**], who shows that two quasi-homogeneous generalized curves with analytically equivalent separatrices are analytically equivalent if and only if the projective holonomies of the component of the exceptional divisor where the separatrices lie are also analytically conjugated. We expect to extend this result to dimension three, thanks to the characterization given in this paper.

2. Pre-simple and simple singularities

Consider a germ of codimension one holomorphic foliation $\mathcal{F}$ defined in a neighbourhood of a point P on a complex manifold M of dimension n by an integrable 1-form ω, and let E be a germ of normal crossings divisor through P, invariant by $\mathcal{F}$.

We will call the *dimensional type* of $\mathcal{F}$ at P, $t = t(\mathcal{F}, P)$, the minimum number of variables needed to write a generator of $\mathcal{F}$. If $\Theta_{M,P}$ denotes the set of germs of holomorphic vector fields in a neighbourhood of P, and

$$\mathcal{D}(\omega)(P) = \{\mathcal{D}(P) \mid \mathcal{D} \in \Omega_{M,P},\ \omega(\mathcal{D}) \equiv 0\}$$

then $t(\mathcal{F}, P)$ equals the codimension of the complex vector space $\mathcal{D}(\omega)(P)$. Note that the integrability of ω implies that the set of vector fields in the kernel of ω define an integrable distribution, and Frobenius' Theorem allows to assume that they are coordinates.

Denote $e = e(E, P)$ the number of components of E through P. Assuming that $E = \prod_{i=1}^{e} x_i$, we can write

$$\omega = \prod_{i=1}^{e} x_i \cdot \left[\sum_{i=1}^{e} a_i \frac{dx_i}{x_i} + \sum_{i=e+1}^{t} a_i dx_i\right].$$

The adapted order of ω at P, relative to E, is then

$$\nu(\mathcal{F}, E; P) = \min\{\nu_P(a_i) \mid 1 \leq i \leq t\},$$

where ν_P denotes the order at P. The adapted multiplicity is

$$\mu(\mathcal{F}, E; P) = \min\{\nu_P(a_i)\}_{1 \leq i \leq e} \cup \{\nu_P(a_i) + 1\}_{i > e}.$$

Then F. Cano and D. Cerveau [**6**] propose the following definition:

Definition 2.1. — *A singularity $P \in \mathrm{Sing}(\mathscr{F})$ is called a pre-simple singularity adapted to E if one of the following holds:*

1. $\nu(\mathscr{F}, E; P) = 0$.
2. $\nu(\mathscr{F}, E; P) = \mu(\mathscr{F}, E; P) = 1$, *and perhaps after performing a change of variables, there exists $s > e$ such that, if $i \leq e$, $a_i = \lambda_i x_s + h.o.t.$, with some $\lambda_i \neq 0$.*

Remark 2.2. — Due to the integrability of ω, in 2. it is sufficient to assume that, for some $i \leq e$, $a_i = x_s + h.o.t.$. The condition above follows.

Pre-simple singularities of dimensional type t are formally conjugated to one of the following meromorphic models, according to [**5, 6, 4**]:

a. $\omega = \sum_{i=1}^{t} \lambda_i \frac{dx_i}{x_i}$, $\lambda_i \in \mathbb{C}^*$.

b. $\omega = \sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \varphi(\mathbf{x}^{\mathbf{P}}) \cdot \sum_{i=2}^{t} \lambda_i \frac{dx_i}{x_i}$, where $p_i \in \mathbb{Z}_{>0}$, $\varphi \in \mathbb{C}[[T]]$, $\varphi(0) = 0$, $\alpha_i \in \mathbb{C}$, $\sum_{i=2}^{t} |\alpha_i| \neq 0$, $\alpha_i \neq 0$ if $i > k$, $\mathbf{x}^{\mathbf{P}} := x_i^{p_1} \cdots x_k^{p_k}$.

c. $\omega = dx_1 - x_1 \sum_{i=2}^{k} p_i \frac{dx_i}{x_i} + x_2^{p_2} \cdots x_n^{p_n} \sum_{i=2}^{t} \lambda_i \frac{dx_i}{x_i}$.

A pre-simple singularity is called *simple* or *reduced* if it is formally conjugated to the types (a) and (b) above, and moreover, the vector $\lambda = (\lambda_1, \ldots, \lambda_t)$ in case (a) or $\lambda = (\lambda_{k+1}, \ldots, \lambda_t)$ in case (b) is strongly non-resonant, i.e., for any non zero function $\Phi : \{1, \ldots, t\} \to \mathbb{Z}_{\geq 0}$, we have that $\sum \Phi(i)\lambda_i \neq 0$. This last condition, that does not appear in [**6**], is imposed in order to avoid dicriticalness.

If P is a pre-simple singularity of dimensional type t, consider the set

$$\{Q \in U \setminus \{P\} \mid Q \in \mathrm{Sing}(\mathscr{F}),\ U \text{ a neighbourhood of } P\}.$$

All elements of this set are singularities of dimensional type 2 (see [**6**] for a proof). Then, P is simple if and only if all elements in the previous set are simple. For instance, in dimension three we have the following result.

***Theorem 2.3* (Cano-Cerveau [6])**. — *Let P be a pre-simple singularity of a singular foliation $\mathscr{F}$ adapted to a normal crossings divisor E over an ambient space of dimension three.*

1. *If $e(E, P) = 2$, take an inmersion $i : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^3, 0)$ through P, tranversal to E. Then P is simple if and only if $i^*\mathscr{F}$ is simple.*
2. *If $e(E, P) = 3$, and there exists an open set $U \ni P$, such that if $Q \neq P$ is a singular point for $\mathscr{F}$ $(e(E, Q) = 2)$, the restriction at $\mathscr{F}$ over a transversal plane through Q is a simple foliation, then $\mathscr{F}$ is simple.*

At this point, let us recall the following result:

***Theorem 2.4* (Cerveau-Mattei [11])**. — *Let $\mathscr{F}$ be a foliation in $(\mathbb{C}^n, 0)$ defined by an integrable 1-form ω. Suppose we have an inmersion $i : (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^n, 0)$, such that 0 is an isolated singularity of $i^*\omega$, reduced. Then, we have the following possibilities:*

1. *If $\mathrm{cod}(\mathrm{Sing}(\omega)) \geq 3$, then ω has an holomorphic first integral.*

2. *If* $\operatorname{cod}(\operatorname{Sing}(\omega)) = 2$*, then* $\mathcal{F}$ *is a cylinder over the foliation induced by* $i^*\omega$*, i.e. if in appropriate coordinates* i *is defined by* $i(x_1, x_2) = (x_1, x_2, \ldots, x_n)$*,* $\mathcal{F}$ *is the pull-back of* $i^*\mathcal{F}$ *by the projection* $\pi : \mathbb{C}^n \to \mathbb{C}^2$ *given by* $\pi(x_1, x_2, \ldots, x_n) = (x_1, x_2)$*. In particular, the dimensional type of* ω *is 2.*

Applying this result to a simple singularity P in $\mathbb{C}^3$, with $e(E, P) = 3$, we obtain that, around nearby singularities, the foliation is simple of dimensional type 2, and trivial over a transversal.

Recall that, in dimension two, there are two types of simple singularities, one of them called saddle-nodes, namely the ones that have a null eigenvalue. This is also valid for foliations of dimensional type two in greater dimensions. If we are in an ambient space of dimension three, and P is a simple singular point for $\mathcal{F}$ with $t(\mathcal{F}, P) = 3$, we say that P is a saddle-node if some of nearby singularities are saddle-nodes. This occurs if the foliation is formally conjugated to the form (b) above, with $k < 3$.

3. Generalized surfaces

From now on, we will work in dimension three. At present, it is not known a result of reduction of singularities for foliations in dimension greater that three, but with that result in hand, our conclusions could be extended to upper dimensions.

Definition 3.1. — *Let* $\mathcal{F}$ *be a holomorphic foliation of codimension one, defined by an integrable 1-form* ω *in* $(\mathbb{C}^3, 0)$*. We will say that* $\mathcal{F}$ *is a generalized surface if in the reduction of singularities of* $\mathcal{F}$*, no saddle-nodes appear, and moreover, it is not dicritical.*

The main objective of this work is the proof of the theorem 1.1.

We have two possible situations. If $\operatorname{cod}(\operatorname{Sing}(\mathcal{F})) = 3$, then $\mathcal{F}$ has a first integral, according to Malgrange's singular Frobenius Theorem [**14**] and the result is obvious. So, in the sequel, we asume that $\operatorname{cod}(\operatorname{Sing}(\mathcal{F})) = 2$.

D. Cerveau introduces in [**10**] a notion of quasi-regular foliation. As a consequence of the results that will follow, our notion of generalized surface agrees with the notion introduced by Cerveau.

Remark 3.2. — In dimension two, a foliation admits only one minimal reduction of singularities. The algorithm is obvious: if one point is singular, blow-up. In dimension three, as in the case of surface, different reduction processes may give as a result desingularized foliations. So, the process is not canonical and we must check that the definition above is independent of the reduction.

The independence of the dicriticalness condition may be seen in [**3**]. Let us also observe that saddle-nodes cannot be destroyed by subsequent blow-ups. So, if we have two different reduction processes over a neighbourhood of 0, such that a saddle-node for $\mathcal{F}_1 = \pi_1^*\mathcal{F}$ appears, take a tranversal germ of surface $S_1 \subseteq \tilde{M}_1$, such that $\mathcal{F}_1|_{S_1}$