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THE GALOISIAN ENVELOPE OF A GERM OF FOLIATION: THE QUASI-HOMOGENEOUS CASE

by

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À José-Manuel, pour ses 60 ans

Abstract. — We give geometric and algorithmic criterions in order to have a proper Galois envelope for a germ of quasi-homogeneous foliation in an ambient space of dimension two. We recall this notion recently introduced by B. Malgrange, and describe the Galois envelope of a group of germs of analytic diffeomorphisms. The geometric criterions are obtained from transverse analytic invariants, whereas the algorithmic ones make use of formal normal forms.

Résumé **(L'enveloppe galoisienne d'un germe de feuilletage : le cas quasi-homogène)**

Nous donnons des critères géométriques et algorithmiques pour qu'un feuilletage quasi-homogène en dimension deux possède une enveloppe galoisienne propre. Nous rappelons cette notion récemment introduite par B. Malgrange et nous décrivons l'enveloppe galoisienne d'un groupe de germes de difféomorphismes analytiques. Les critères géométriques sont obtenus à partir d'invariants analytiques transverses, tandis que les critères algorithmiques utilisent les formes normales.

Introduction

There are several notions of integrability for a system of differential equations. Most of them are related to the existence of a sufficient number of first integrals for the solutions of the system. These definitions differ each other on the additional properties required for this family of invariants functions. We can separate them into two types:
— conditions between the first integrals: one may ask commutativity conditions for the Poisson bracket, or relax such a condition;
— conditions on the nature of these functions: rational, meromorphic or multivalued

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functions in some "reasonable" class of transcendency.
The main methods for proving non integrability (analytical methods, Ziglin method or Morales-Ramis method) are based on the linearization of the system around a particular solution. Therefore they only deliver sufficient criterions on non integrability, using for the last mentioned method *linear* differential Galois theory.

In order to investigate the second type of condition, and –in the future– to get necessary and sufficient conditions for integrability, we have to consider the system in the whole, which suggests to consider a *non linear* differential Galois theory. The first attempts in this direction was done by J. Drach and E. Vessiot. More recently, B. Malgrange introduced in [**12**] (see also the introductive version [**13**]) a "Galois envelope" for any dynamical system, namely the smallest D-groupoid which contains the solutions of the system. Roughly speaking, a D-groupoid is a system of partial differential equations whose local solutions satisfy groupoid conditions outside an analytic codimension one set. They are not strict Lie groupoid, in order to deal with singular systems. As a matter of introduction to this notion, we shall describe in the first section the Galois envelope of a group of germs of analytic diffeomorphisms at the origin of $\mathbb{C}$.

Each D-groupoid admits a D-algebra obtained by the linearization of its equations along the identity solutions. The local solutions of this linear differential system are stable under the Lie bracket outside of a codimension one analytic set. The Galois envelope of a singular analytic foliation $\mathcal{F}$ is the smallest D-groupoid $\mathrm{Gal}(\mathcal{F})$ whose D-algebra contains the germs of tangent vector fields to $\mathcal{F}$. It is a proper one if it doesn't coincide with the whole groupoid $\mathrm{Aut}(\mathcal{F})$ obtained by writing the equations of invariance of the foliation under a local diffeomorphism. In this case –which is not the general case–, its solutions satisfy an additional differential relation, and we shall say that the foliation is Galois reducible.

For a local codimension one singular foliation defined by a holomorphic one-form ω, this reducibility property is equivalent to the existence of a Godbillon-Vey sequence of finite length for ω (at most three): there exists a finite sequence of meromorphic one forms ω_0, ω_1, and ω_2 such that ω_0 is an equation of the foliation and

$$d\omega_0 = \omega_0 \wedge \omega_1,\ d\omega_1 = \omega_0 \wedge \omega_2,\ d\omega_2 = \omega_1 \wedge \omega_2.$$

This fact was described in a manuscript of B. Malgrange [**14**], and then has been extensively proved by G. Casale in [**5**] with some different arguments. In particular, the transverse rank of $\mathrm{Gal}\mathcal{F}$ (i.e. the order of its transverse local expression) is also the minimal lenght of a Godbillon-Vey sequence for $\mathcal{F}$. Finally, G. Casale proved in [**2**] that this Godbillon-Vey condition is also equivalent to the existence of first integrals for the foliation with a particular type of transcendency which belongs to a Darboux or Liouville or Riccati type differential extension, according to the transverse rank of the Galois envelope. These different points of view on the Galois reducibility admit a generalization for higher codimension foliations: see [**6**] for the Painlevé 1 foliation.

In the present paper we shall only deal with codimension one foliations in ambient spaces of dimension two. Therefore, we expect the existence of at most one first integral, and we only have to discuss the second type of integrability condition: the existence of such a first integral in a given class of transcendency. The previous discussion allows us to reformulate the integrability problem as following: *give necessary and sufficient criterions for the Galois reducibility of a germ of codimension one foliation.* We present an answer to this problem in the following context: $\mathcal{F}$ is defined by a vector field $X = X_h + \cdots$ where the "initial" hamiltonian vector field

$$X_h = \frac{\partial h}{\partial y}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial y}$$

is quasi-homogeneous with to respect to $R = p_1 x\frac{\partial}{\partial x} + p_2 y\frac{\partial}{\partial y}$ (p_1, p_2 positive integers): $R(h) = \delta h$, $\delta = \deg_R(h)$. The dots means terms of higher quasihomogeneous degree. We furthermore require that h has an isolated singularity (with Milnor number μ) and that X still keep invariant the analytic set $h = 0$. Therefore, X is a logarithmic vector field for the polar set $h = 0$, and we have:

$$X = aX_h + bR,\ a \in \mathcal{O}_2, b \in \mathcal{O}_2,\ a(0) = 1$$

with $deg_R(bR) > \deg_R(X_h)$. The restriction to this class of foliation is motivated by the two following reasons:

— the desingularization of these foliations by blowing up's is "simple": it is similar to the one of the quasi-homogeneous function h: the exceptional divisor is only a chain of projective lines and all the pull-back of the irreducible components of h –excepted the axis if they appear in h– meet the same "principal" projective line C.

— in this class of foliations, we have at our disposal *formal normal forms* which give us complete formal invariants: see **[21]**.

This will allow us to give two different types of criterions for the Galois reducibility of $\mathcal{F}$: a geometric one which is related to the holonomy of the principal component C of the desingularized foliation, and an algorithmic one which directly holds on the normalized formal equation of the foliation. For the first one, let us denote $\mathrm{Hol}(\mathcal{F})$ the holonomy group of the principal component C for the desingularized foliation. This is an analytic invariant of $\mathcal{F}$ (in fact, this "transverse invariant" is also a complete invariant in this quasi-homogeneous context: see **[8]**). We prove in theorem (2.4) the following result :

Theorem 1. *The Galois groupoid of the germ of quasi-homogeneous foliation $\mathcal{F}$ is a proper one if and only if the Galois envelope of* $\mathrm{Hol}(\mathcal{F})$ *is a proper one.*

This theorem reduces the initial problem to the determination of the Galois envelope of a subgroup G of Diff($\mathbb{C}$,0), which is described in the first section (theorem 1.8). The main argument in the proof of this theorem is an extension of the equation which define the Galois envelope of $\mathrm{Hol}(\mathcal{F})$ to the whole exceptional divisor. This is possible, since the elements of the holonomy group of C are solutions of this equation and therefore keep it invariant. This proof suggests that even in non quasi-homogeneous cases, these criterions for the Galois reducibility will only depend on the transverse structure of the foliation.

Theorem 1 is not an explicit criterion since in general, we can't compute the invariant $\mathrm{Hol}(\mathcal{F})$. In order to get an algorithmic criterion, we recall in section 3 the formal normal forms for this class of foliations. Notice that in general these models are divergent models. The radial component of these normal forms make appear a collection $\mathcal{L}(\mathcal{F})$ of μ formal one-variable vector fields, and it turns out that this collection (up to a commun conjugacy) is a complete invariant for the formal class of $\mathcal{F}$. It must be surprising to try to charaterize the Galois reducibility of $\mathcal{F}$ using only formal invariants. Nevertheless, we can perform it according to the two following facts:
— if a foliation is Galois reducible, then its formal normal form is a convergent one;
— if the foliation $\mathcal{F}$ is a "non exceptional" one (see **[7]**), then there exists a convergent conjugacy between $\mathcal{F}$ and its model.
Clearly, for exceptional foliations, we need an additional condition on the analytic class of $\mathcal{F}$, which is not yet an algorithmic one. The central result of this work is the following theorem which summarize theorem 3.5, corollary 3.7 and theorem 3.8:

Theorem 2. *If the quasi-homogeneous foliation $\mathcal{F}$ is a non exceptional one, the Galois envelope of $\mathcal{F}$ is proper if and only if the explicit invariant $\mathcal{L}(\mathcal{F})$ generates a finite dimensional Lie algebra. In this case, this one is always of dimension one, and the foliation is at most Liouvillian.*

If the quasi-homogeneous foliation $\mathcal{F}$ is an exceptional one, the Galois envelope of $\mathcal{F}$ is proper if and only if the explicit invariant $\mathcal{L}(\mathcal{F})$ is a finite dimensional Lie algebra, and the analytic invariants of $\mathcal{F}$ are of "unitary" or "binary" type. In this case, the foliation will be a Liouvillian one (for unitary invariants), or of Riccati type, (for binary invariants).

We shall recall in the first section the definition of unitary or binary invariants which is a terminology introduced by J. Ecalle. The first part of the theorem is an extension of a result of F. Loray and R. Meziani for nilpotent singularities **[11]**, while the second one is an extension of a theorem of G. Casale for reduced singularities **[5]**. Notice that in the local context, the Galois reducible foliations which are not Liouvillian are very rare.

Clearly, the relationship between the algorithmic invariant $\mathcal{L}(\mathcal{F})$ and the geometric one $\mathrm{Hol}(\mathcal{F})$ has a transcendental nature since the first one is directly obtained from the differential equation whereas the second one is related to the solutions of this equation. Nevertheless, for Galois reducible foliations we can describe this relationship: it reduces to the exponential map of the one-variable vector fields of $\mathcal{L}(\mathcal{F})$. In order to check this fact it is more convenient to consider an equivalent data to $\mathrm{Hol}(\mathcal{F})$: the relative holonomy of $\mathcal{F}$ with respect to its initial part defined by X_h (see section 4).

Finally, we conclude this paper with a list of open questions related to the present results.

1. The Galois envelope of a subgroup of Diff($\mathbb{C}$,0)

Let Δ be a disc around 0 in $\mathbb{C}$. We first recall the list of all the D-groupoids on Δ (see **[14]** and **[3]**). We denote $(x, y, y_1, y_2, \ldots, y_k)$ the coordinates for the space of k-jets of maps from Δ to itself.

Theorem 1.1. — *The differential ideal of a D-groupoid on Δ is generated by a meromorphic equation of one of the five types:*

1. *D-groupoids of order zero: they are generated by an equation of the form: $h(x) - h(y) = 0$ where h is a holomorphic function on Δ. We denote them: $G_0(h)$.*
2. *D-groupoids of order one: they are generated by an equation of the form: $\eta(y)(y_1)^n - \eta(x) = 0$ where n is an integer, and η a meromorphic function on Δ. We denote them $G_1^n(\eta)$.*
3. *D-groupoids of order two: they are generated by an equation of the form: $\mu(y)y_1 + \frac{y_2}{y_1} - \mu(x) = 0$ where μ is meromorphic on Δ. We denote them $G_2(\mu)$.*
4. *D-groupoids of order three: they are generated by an equation of the form: ${\nu(y)y_1}^2 + 2\frac{y_3}{y_1} - 3\left(\frac{y_2}{y_1}\right)^2 - \nu(x) = 0$ where ν is meromorphic on Δ. We denote them $G_3(\nu)$.*
5. *The D-groupoid of infinite order G_∞ defined by the trivial equation $0 = 0$, whose solutions are the whole sheaf* Aut(Δ).

The Galois envelope of a subgroup G of Diff(C,0) is the smallest D-groupoid in the previous list which admits all the elements g of G as solutions. Clearly, the existence of a proper Galois envelope of finite order k, only depends on the analytic class of G. The Galois envelope for a monogeneous subgroup generated by g is the Galois envelope of g itself, since all the iterates of g will also satisfy the same equation, by composition or inversion stability. The Galois envelope Gal(g) of g is given by the two following results, see B. Malgrange **[14]**, and G. Casale ([**3**]). Let $\alpha = g'(0)$. If α is an irrational number, then g is formally linearizable. We have:

Proposition 1.2. — *A formally linearizable diffeomorphism has a proper Galois envelope if and only if it is an analytically linearizable diffeomorphism. In this case, its Galois envelope is a rank one D-groupoid.*

If α is a rational number, g is a resonant diffeomorphism, and there exists an integer q such that g^q is tangent to the identity. The following lemma

Lemma 1.3 **([3])**. — *For all non vanishing integer q,* $\mathrm{Gal}(g) = \mathrm{Gal}(g^q)$.

reduces the study to the case $\alpha = 1$. Any diffeomorphism tangent to the identity to an order k is conjugated via a formal series to a normal form g_N which is the exponential of the vector field $\frac{x^{k+1}}{1+\lambda x^k}\frac{d}{dx}$. Following the description of J. Martinet and J.P. Ramis,