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FAMILIES OF AUTOMORPHIC FORMS ON DEFINITE QUATERNION ALGEBRAS AND TEITELBAUM'S CONJECTURE

by

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Abstract. — The main goal of this note is to describe a new proof of the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum. This proof relies on Teitelbaum's approach to the $\mathscr{L}$-invariant based on the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves.

***Résumé* (Familles de formes automorphes sur les algèbres quaternioniques et conjecture de Teitelbaum)**

Cet article fournit une nouvelle démonstration de la conjecture de Mazur, Tate et Teitelbaum sur les « zéros exceptionnels » des fonctions L p-adiques. Cette démonstration repose sur une définition de l'invariant $\mathscr{L}$ proposée par Teitelbaum, qui repose sur la théorie de l'uniformisation p-adique des courbes de Shimura.

Introduction

Let $f = \sum a_n q^n$ be a newform of even weight $k_0+2 \geq 2$ on $\Gamma_0(Np)$, where $N \geq 4$ is a positive integer and p is a prime which does not divide N. We denote by $L(f,s)$ the complex L-function attached to f, and by $L(f,\chi,s)$ its twist by a Dirichlet character χ. A theorem of Shimura asserts the existence of a complex period Ω_f such that the special values

$$L(f,\chi,j)/\Omega_f \quad \text{with } 1 \leq j \leq k_0+1$$

belong to the subfield K_f of $\mathbb{C}$ generated by the Fourier coefficients of f, and even to its ring of integers. These special values (when χ ranges over the Dirichlet characters of p-power conductor) can be interpolated p-adically, yielding the Mazur-Swinnerton-Dyer p-adic L-function $L_p(f,s)$, a p-adic analytic function whose definition depends on the choice of Ω_f. Denote by

$$L^*(f,\chi,1+k_0/2) := L(f,\chi,1+k_0/2)/\Omega_f,$$

the *algebraic part* of $L(f,\chi,s)$ at the central critical point $s = 1+k_0/2$.

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The modular form f is said to be *split multiplicative* if

$$f|U_p = p^{k_0/2} f.$$

In that case, $L_p(f, s)$ has a so-called *exceptional zero* at $s = 1+k_0/2$ arising from the p-adic interpolation process. In fact, like its classical counterpart, the p-adic L-function $L_p(f, s)$ has a functional equation of the form

$$L_p(f, k_0 + 2 - s) = \epsilon_p(f)\langle N\rangle^{s-1-k_0/2} L_p(f, s), \tag{1}$$

and the sign $\epsilon_p(f) = \pm 1$ that appears in this equation is related to the the sign $\epsilon_\infty(f)$ in the classical functional equation for $L(f, s)$ by the rule

$$\epsilon_p(f) = \begin{cases} -\epsilon_\infty(f) & \text{if } f \text{ is split multiplicative;} \\ \epsilon_\infty(f) & \text{otherwise.} \end{cases}$$

In the case where f is a split multiplicative newform, Mazur, Tate and Teitelbaum made the following conjecture in [**18**]:

Conjecture 1. — *There exists a constant $\mathscr{L}(f) \in \mathbb{C}_p$, which depends only on the restriction of the Galois representation attached to f to a decomposition group at p, and such that*

$$L'_p(f, \chi, 1 + k_0/2) = \mathscr{L}(f) L^*(f, \chi, 1 + k_0/2), \tag{2}$$

for all χ with $\chi(-1) = \chi(p) = 1$.

The constant $\mathscr{L}(f)$, which Mazur, Tate and Teitelbaum called the *L-invariant*, was only defined in [**18**] in the weight two case $k_0 = 0$. In the higher weight case $k_0 > 0$, several a priori inequivalent definitions of $\mathscr{L}(f)$ were subsequently proposed.

1. In [**23**], Teitelbaum offered the first definition for $\mathscr{L}(f)$. This invariant, denoted $\mathscr{L}_T(f)$, is based on the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves and is only defined for modular forms which are the Jacquet-Langlands lift of a modular form on a Shimura curve uniformized by Drinfeld's p-adic upper half plane. This occurs, for example, when the conductor of f can be written as a product of three pairwise relatively prime integers of the form

$$pN = pN^+N^-,$$

   where N^- is the square-free product of an *odd* number of prime factors. A modular form which satisfies this condition will be said to be *p-adically uniformisable.*
2. Coleman [**5**] then proposed an analogous but more general invariant $\mathscr{L}_C(f)$ by working directly with p-adic integration on the modular curve attached to the group $\Gamma_0(p) \cap \Gamma_1(N)$.
3. Fontaine and Mazur [**17**] gave a definition for the so-called *Fontaine-Mazur $\mathscr{L}$-invariant* $\mathscr{L}_{FM}(f)$ in terms of the filtered, Frobenius monodromy module of the p-adic Galois representation attached to f.

4. In **[19]**, Orton has introduced yet a further $\mathscr{L}$-invariant $\mathscr{L}_O(f)$, based on the group cohomology of arithmetic subgroups of $\mathbf{GL}_2(\mathbb{Z}[1/p])$, extending to forms of higher weight the approach taken in **[12]** for $k_0 = 0$.
5. Finally Breuil defined in **[2]** the $\mathscr{L}$-invariant $\mathscr{L}_{Br}(f)$ in terms of the p-adic representation of $\mathbf{GL}_2(\mathbb{Q}_p)$ attached by him to f.

We now know that all the above $\mathscr{L}$-invariants are equal (when they are defined) as result of work of many people, which we briefly list below (see **[9]** for a more detailed account of these various articles and preprints).

The equality of the $\mathscr{L}$-invariants $\mathscr{L}_C(f)$ and $\mathscr{L}_{FM}(f)$ was proved in **[7]** by making explicit the comparison isomorphism between the p-adic étale cohomology and log-crystalline cohomology of the modular curve $X_0(Np)$ with respective coefficients. The equality of $\mathscr{L}_T(f)$ and $\mathscr{L}_C(f)$ (when they are both defined) was proved in **[16]** by interpreting $\mathscr{L}_T(f)$ as the $\mathscr{L}$-invariant of a filtered, Frobenius monodromy module. Breuil proved in **[2]** the equality $\mathscr{L}_{Br}(f) = \mathscr{L}_O(f)$, which is a manifestation of the local-global compatibility for the p-adic Langlands correspondence.

It was first observed by Greenberg and Stevens for weight two (in **[15]**) and by Stevens in general (in **[22]**) that p-adic deformations of f, i.e. p-adic families of modular eigenforms are relevant for conjecture (1). To describe these objects precisely, let

$$\mathscr{W} := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Q}_p^\times)$$

denote the weight space, viewed as the $\mathbb{Q}_p$-points of a rigid analytic space. There is a natural inclusion $\mathbb{Z} \subset \mathscr{W}$ by sending k to the function $x \mapsto x^k$. Write $A(U)$ for the ring of rigid analytic functions on U, for any affinoid disk $U \subset \mathscr{W}$.

A *p-adic family of eigenforms* interpolating f is the data of a disk U with $k_0 \in U$, and of a formal q-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n q^n, \tag{3}$$

with coefficients in $A(U)$ satisfying:

1. For every $k \in U \cap \mathbb{Z}^{\geq 0}$,
$$f_k := \sum_{n=1}^{\infty} a_n(k) q^n$$
is the q-expansion of a normalized eigenform of weight $k+2$ on the congruence group $\Gamma_1(p) \cap \Gamma_0(N)$;
2. $f_{k_0} = f$.

The existence and essential uniqueness of the family f_∞ interpolating f is proved in **[6]**.

Greenberg and Stevens for weight two and Stevens in general first proved that $\mathscr{L}_C(f) = -2(\mathrm{dlog} a_p)_{\kappa=k_0}$. Colmez generalized the Galois cohomology calculations in **[15]** by working inside Fontaine's rings and proved the equality $\mathscr{L}_{FM}(f) = -2(\mathrm{dlog} a_p)_{\kappa=k_0}$ in **[11]**. He also proved the equality $\mathscr{L}_{Br}(f) = -2(\mathrm{dlog} a_p)_{\kappa=k_0}$ in

[**10**] by using the p-adic local Langlands correspondence for trianguline representations. Let us remark that in fact the quantity $\mathscr{L}(f)_{NoName} := -2(\mathrm{dlog} a_p)_{\kappa=k_0}$ behaves like an $\mathscr{L}$-invariant: it satisfies the equation (2) of conjecture 1 (see [**22**]) and it is a local invariant of f in the sense that it is invariant to twists of f by Dirichlet characters trivial at p (in fact it is invariant to all twists by Dirichlet characters.)

Conjecture (1) was first proved in [**15**] for weight two, and several different proofs have been announced in the higher weight case:

1. By Kato-Kurihara-Tsuji, working with the invariant $\mathscr{L}_{FM}(f)$;
2. By Glenn Stevens, working with $\mathscr{L}_C(f)$;
3. By Orton, working with $\mathscr{L}_O(f)$ in [**19**];
4. By Emerton working with $\mathscr{L}_{Br}(f)$ in [**14**].

The first two proofs are still unpublished but an account of the approach of Kato-Kurihara-Tsuji can be found in [**8**] while Stevens gave a series of lectures on his theory during the *Automorphic Forms* semester in Paris, 1998. Notes to these lectures, to which we will refer as [**22**], although not yet published circulated widely in the mathematical community and greatly influenced articles like [**3**], [**4**] and the present note. As these notes have not been published we will sketch proofs of all the results quoted from them.

The main goal of this note is to describe a new proof of Conjecture 1 which applies to forms which are p-adically uniformisable.

Theorem 2. — *Assume that f is p-adically uniformisable. Then*

$$L_p'(f,\chi,1+k_0/2) = \mathscr{L}_T(f) L^*(f,\chi,1+k_0/2), \tag{4}$$

for all Dirichlet characters χ satisfying $\chi(-1)=\chi(p)=1$.

Our proof of Theorem 2 is based on Teitelbaum's definition of the L-invariant: this is why it needs to be assumed that f is p-adically uniformisable. Thus the Ceřednik-Drinfeld theory of p-adic uniformisation of Shimura curves and the Jacquet-Langlands correspondence, which play no role in the earlier proofs of Stevens and Kato-Kurihara-Tsuji, are key ingredients in our approach. Section 1 supplies the necessary definitions concerning automorphic forms on definite quaternion algebras, and Section 2 recalls a few basic facts concerning p-adic integration on Shimura curves, including Teitelbaum's theory of the "p-adic Poisson kernel" and his definition of the invariant $\mathscr{L}_T(f)$.

Guided by the Jacquet-Langlands correspondence between classical modular forms and automorphic forms on quaternion algebras, Section 3 describes a theory of p-adic families of automorphic forms on definite quaternion algebras, based on ideas of Stevens, Buzzard and Chenevier. The resulting structures are used to prove the following theorem in Section 4, which relates Teitelbaum's L-invariant to the derivative of the Fourier coefficient $a_p(k)$ with respect to k.

Theorem 3. — *Suppose that f is p-adically uniformisable. Then*

$$\mathscr{L}_T(f) = -2\mathrm{dlog}(a_p)_{\kappa=k_0}. \tag{5}$$

The ideas of Orton in [**19**], which are recalled in Section 5, make it apparent that the definition of the invariants $\mathscr{L}_T(f)$ and $\mathscr{L}_O(f)$ are very similar in flavour. The calculations of Sections 1 to 4, when transposed to the context of a modular form on $\mathbf{GL}_2(\mathbb{Q})$, with the "integration on $\mathscr{H}_p \times \mathscr{H}$" defined in terms of modular symbols playing the role of the p-adic line integrals on Drinfeld's upper half-plane, leads to the proof of the following analogue of Theorem 3, which is described in Section 6:

Theorem 4. — *Let f be a modular form of weight k on $\Gamma_0(N)$ which is split multiplicative at p. Then*

$$\mathscr{L}_O(f) = -2\mathrm{dlog}(a_p)_{\kappa=k_0}. \tag{6}$$

Theorem 2 now follows directly from Theorems 3 and 4, in light of Orton's proof of Conjecture (1).

The remainder of the text will focus on explaining the proofs of Theorems 3 and 4, which are independent (both in their statement, and their formulation) of the existence and basic properties of either the p-adic L-function or the p-adic Galois representation attached to f and the Coleman family interpolating it.

We emphasize that the proof of Theorem 2 owes much to the ideas that are already present in the earlier (although still unpublished) approaches of Stevens and Kato-Kurihara-Tsuji. The main virtues (and drawbacks) of our method are inherently the same as those in Teitelbaum's approach to defining the $\mathscr{L}$-invariant: a gain in simplicity (because the method involves p-adic integration on a Mumford curve rather than a modular curve, and requires no information about Galois representations) offset by a certain loss of generality (since the method only applies to automorphic forms that can be obtained as the Jacquet-Langlands lift of a modular form on a p-adically uniformized Shimura curve). A second, less immediately apparent advantage of our approach lies in the insights arising from the connection that is drawn between the two-variable p-adic L-function $L_p(k,s)$ attached to f_∞ and the p-adic uniformisation of Shimura curves. In particular, the new ideas introduced in this article form the basis for the proof of the main result of [**1**], which, in the case where f corresponds to a modular elliptic curve E over $\mathbb{Q}$ and $\epsilon_\infty(f) = -\epsilon_p(f) = -1$, relates the leading term of $L_p(k,s)$ at the central critical point $(k,s) = (2,1)$ to the formal group logarithm of a global point on $E(\mathbb{Q})$.

1. Automorphic forms on quaternion algebras

Suppose from now on that f is p-adically uniformisable, so that its level pN can be factored as

$$pN = pN^+N^-, \text{ where } \gcd(N^+, N^-) = 1, \tag{7}$$

and where N^- is square-free and has an *odd* number of prime factors. Let B denote the quaternion algebra over $\mathbb{Q}$ ramified exactly at $N^-\infty$, and let $\mathscr{R}$ denote a maximal