# RIGIDITY OF FIBRATIONS 

## by

Jorge Vitório Pereira \& Paulo Sad

To José Manuel Aroca in occasion of his $60^{\text {th }}$ birthday


#### Abstract

We consider a set $\Gamma$ of points in the projective plane obtained as the intersection of two curves of the same degree. We blow-up the projective plane at that points to get $S_{\Gamma}$. We consider the foliation $\mathcal{F}_{\gamma}$ in $S_{\Gamma}$ obtained from the pencil of the two curves above. Under generic conditions $\mathcal{F}_{\gamma}$ is isolated in the space of foliations of $S_{\Gamma}$. Résumé (Rigidité des fibrations). - Nous considérons l'ensemble $\Gamma$ des points du plan projectif obtenu comme intersection de deux courbes du même degré. Nous éclatons cet ensemble pour obtenir la surface $S_{\Gamma}$ et nous considérons sur $S_{\Gamma}$ le feuilletage $\mathscr{F}_{\gamma}$ obtenu à partir du pinceau de deux courbes précédentes. Sous des conditions de généricité $\mathcal{F}_{\gamma}$ est isolé dans l'espace des feuilletages de $S_{\Gamma}$.


## 1. Introduction

Let $\Gamma$ be a finite set of points in the projective plane $\mathbb{P}^{2}$ defined as the intersection of two transverse curves of the same degree (we say that $\Gamma$ is a complete intersection set); let also $\pi: S_{\Gamma} \rightarrow \mathbb{P}^{2}$ be the blow-up of $\mathbb{P}^{2}$ at the points of $\Gamma$. The surface $S_{\Gamma}$ admits a natural foliation $\mathscr{G}_{\Gamma}$ : the strict transform of the pencil $\mathscr{F}_{\Gamma}: F d G-G d F=0$ generated by the curves $\{F=0\}$ and $\{G=0\}$ that define $\Gamma$.

A natural problem is to understand the families of reduced foliations of surfaces (in the sense of $[2]$ ) containing ( $S_{\Gamma}, \mathscr{G}_{\Gamma}$ ); this is related to studying the foliations of $\mathbb{P}^{2}$, in a neighborhood of $\mathscr{F}_{\Gamma}$, that have radial singularities close to the points of $\Gamma$.

We consider in this paper the particular situation where the surface $S_{\Gamma}$ does not change in the family (or, equivalently, we look at the foliations of $\mathbb{P}^{2}$ with radial

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singularities at the points of $\Gamma$ ). The leaves of $\mathscr{G}_{\Gamma}$ are fibers of the holomorphic fibration $(F / G) \circ \pi \rightarrow \mathbb{P}^{1}$. In order to study a deformation $\mathscr{F}$ of this fibration (in the space of foliations of $S_{\Gamma}$ ) we analyze how a generic fiber $\widetilde{C}$ of $\mathscr{G}{ }_{\Gamma}$ is intersected by t he leaves of $\mathscr{G}$. If $\widetilde{C}$ is not $\mathscr{G}$-invariant then $\mathcal{N}_{g} \cdot \widetilde{C}=\operatorname{tang}(\mathscr{G}, \widetilde{C})+\chi(\widetilde{C})$, where $\mathcal{N}_{\mathscr{G}}$ is the normal bundle of $\mathscr{G}, \chi(\widetilde{C})$ is the Euler characteristic of $\widetilde{C}$ and $\operatorname{tang}(\mathscr{G}, \widetilde{C})$ is the number of tangency points between $\mathscr{G}$ and $\widetilde{C}$. Notice that $\operatorname{tang}(\mathscr{G}, \widetilde{C}) \geq 0$ and
 $\widetilde{C} \cdot \widetilde{C}$, where $Z(\mathscr{G}, \widetilde{C})$ denotes the number of singularities of $\mathscr{G}_{\Gamma}$ along $\widetilde{C}$, and we get that $\operatorname{tang}(\mathscr{F}, \widetilde{C})=-\chi(\widetilde{C})$.

Let $c \in \mathbb{N}$ be the common degree of the polynomials $F$ and $G$. When $c=1$ or $c=2$ we have $\chi(\widetilde{C})=2$ and we get a contradiction unless $\mathscr{G}=\mathscr{G}_{\Gamma}([\mathbf{1 2}])$. When $c=3$ we have $\operatorname{tang}(\mathscr{G}, \widetilde{C})=-\chi(\widetilde{C})=0$ and therefore $\mathscr{G}$ is transverse to the generic fiber of $\mathscr{G}_{\Gamma}$, implying that the regular fibers are all isomorphic; this is not possible for a generic choice of $F$ and $G$, and we conclude again that $G=G_{\Gamma}$ in this case (see [11] for a related result). When $c \geq 4$ this type of argument fails since $\chi(\widetilde{C})<0$. Nevertheless we are able to prove for $c \geq 3$ :

Theorem 1. - If $\Gamma$ is a generic complete intersection set then $\mathcal{F}_{\Gamma}$ is an isolated point of the space of foliations of $S_{\Gamma}$, i.e., $\mathcal{F}_{\Gamma}$ is rigid.

In the statement generic complete intersection set refers to the set of base points of a generic element of the space of lines of $\mathbb{P H}^{0}\left(\mathbb{P}^{2}, \vartheta_{\mathbb{P}^{2}}(c)\right)$; in other words, the couple $(F, G)$ of polynomials of degree $c \in \mathbb{N}$ is generically chosen in order to define $\Gamma$. In $\S 3.2$ we exhibit some examples of non-rigidity to show that the hypothesis of genericity is necessary.

We have no result when the surface $S_{\Gamma}$ changes in the family of reduced foliations; but still we should mention that for $c=3$ we can only deform $\mathscr{G}_{\Gamma}$ as a fibration (starting with a generic choice of $\Gamma$ ). In fact, $\mathscr{G}_{\Gamma}$ has Kodaira dimension equal to 1 and this dimension is constant along the family ([2]). We then apply the Classification Theorem ([1]) to conclude that any foliation in the family is an elliptic fibration.

The proof of Theorem 1 relies on the analysis of the indexes of a plane foliation along a smooth invariant algebraic curve. Let $\{F=0\}$ be such a curve, of degree $c \in \mathbb{N}$, containing singularities of the foliation at the intersection points with another curve $\{G=0\}$ of degree $k \leq c$. We prove then that if $(F, G)$ is generically chosen the set of indexes is sufficient to identify completely the foliation (Theorem 2.2). Application of this result in order to prove Theorem 1 is not immediate; we have to show first that the defining curves for the set $\Gamma$ are invariant curves of the foliation.

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## 2. Variation of Indexes

2.1. Division Lemma. - All foliations, unless stated otherwise, are supposed to have isolated singularities.

Let $C \subset \mathbb{P}^{2}$ be a smooth curve of degree $c \in \mathbb{N}$, invariant by a plane projective foliation $\mathcal{F} \in \operatorname{Fol}(d)$ of degree $d \in \mathbb{N}$. The Lemma below can be implicitly found in [4, Proof of Proposition 3]; we assume that $\mathcal{F}$ is defined by $\omega=0$, $\omega$ a homogeneous 1-form of $\mathbb{C}^{3}$ of degree $d+1$ (or by a homogeneous vector field of $\mathbb{C}^{3}$ of degree $d \in \mathbb{N}$ ), and that $C$ is defined by $F=0, F$ a homogeneous polynomial of degree $c \in \mathbb{N}$. Let us denote by $R$ the radial vector field of $\mathbb{C}^{3}$.

Lemma 2.1. - There exist a polynomial $G$ of degree $d-c+2$ and a 1-form $\eta$ of degree $d-c+1$, both homogeneous, such that

$$
\omega=G d F-\frac{\operatorname{deg}(F)}{\operatorname{deg}(G)} F d G+F \eta \quad \text { and } i_{R}(\eta)=0
$$

Furthermore, the foliation $\mathcal{F}_{\eta}$ defined by $\eta=0$ depends only on $\mathcal{F}$ and $C$ when $d \leq 2 c-2$.

Proof. - It follows from ([4, Proposition 1]) that there exist a homogeneous polynomial $G$ of degree $d-c+2$ and a homogeneous 1-form $\alpha$ of degree $d-c+1$ such that
(1)

$$
\omega=G d F+F \alpha
$$

After contracting the above expression with the radial vector field we obtain

$$
\operatorname{deg}(F) F G+F i_{R} \alpha=0
$$

We rewrite (1) as

$$
\omega=G d F-\frac{\operatorname{deg}(F)}{\operatorname{deg}(G)} F d G+F\left(\alpha+\frac{\operatorname{deg}(F)}{\operatorname{deg}(G)} d G\right)
$$

and define $\eta:=\alpha+\frac{\operatorname{deg}(F)}{\operatorname{deg}(G)} d G$; it follows that $i_{R}(\eta)=0$.
Let us replace (1) by $\omega^{\prime}=G^{\prime} d F^{\prime}+F^{\prime} \alpha^{\prime}$, where $\omega^{\prime}=\lambda \omega$ and $F^{\prime}=\mu F$ for $\lambda, \mu \in \mathbb{C}$. Consequently:

$$
\omega=\left(\frac{\mu}{\lambda} G^{\prime}\right) d F+F\left(\frac{\mu}{\lambda} \alpha^{\prime}\right)=G d F+F \alpha
$$

and

$$
\left(\frac{\mu}{\lambda} G^{\prime}-G\right) d F=F\left(\alpha-\frac{\mu}{\lambda} \alpha^{\prime}\right)
$$

From $\left(\frac{\mu}{\lambda} G^{\prime}-G\right)_{\mid C} \equiv 0$ we have $\frac{\mu}{\lambda} G^{\prime}-G=P . F$ for some homogeneous polynomial $P$; two possibilities arise:

- $d<2 c-2$; therefore $\frac{\mu}{\lambda} G^{\prime}=G, \frac{\mu}{\lambda} \alpha^{\prime}=\alpha$ and we get

$$
\eta^{\prime}=\alpha^{\prime}+\frac{\operatorname{deg}(F)}{\operatorname{deg}(G)} d G^{\prime}=\frac{\mu}{\lambda} \eta
$$

- $d=2 c-2$, so that $\frac{\mu}{\lambda} G^{\prime}-G=a F, \alpha-\frac{\mu}{\lambda} \alpha^{\prime}=a d F$ for $a \in \mathbb{C}$. It follows that $\alpha-\frac{\mu}{\lambda} \alpha^{\prime}=\frac{\mu}{\lambda} d G^{\prime}-d G$ and again $\eta^{\prime}=\frac{\mu}{\lambda} \eta$.

We observe that $\mathcal{F}_{\eta}$ may have a curve of singularities.
Our results follow from the analysis of the behavior of $\mathscr{F}_{\eta}$ with respect to $C$ when $d \leq 2 c-2$. For the moment we remark that:

- the singularities of $\mathcal{F}$ contained in $C$ are the points of $\{G=0\} \cap C$.
- $C$ is not contained in the singular set of $\mathcal{F}_{\eta}$ (because $\left.\operatorname{deg}(\eta)<\operatorname{deg}(F)\right)$.
- $C$ is not $\mathscr{F}_{\eta}$-invariant (because otherwise $\operatorname{deg}(C) \leq \operatorname{deg}\left(\mathscr{F}_{\eta}\right)+1$, see [4], or $c \leq d-c+1$ ). Let us write $k=\operatorname{deg}(G)=d-c+2$ for simplicity, so that $\operatorname{deg}\left(\mathscr{F}_{\eta}\right)=k-2$. Since $\operatorname{tang}\left(\mathscr{F}_{\eta}, C\right)=\mathcal{N}_{\mathscr{F}_{\eta}} . C-\chi(C)=k . c-\left(2-2 \frac{(c-1)(c-2)}{2}\right)$, we find $\operatorname{tang}\left(\mathcal{F}_{\eta}, C\right)=c(k+c-3)$; the tangency points between $C$ and $\mathcal{F}_{\eta}$ are given by the common solutions of $F=0$ and $d F\left(Z_{\eta}\right)=0\left(Z_{\eta}\right.$ is the homogeneous vector field of $\mathbb{C}^{3}$ of degree $k-2$ which defines $\mathcal{F}_{\eta}$ ).
2.2. Indexes and Foliations. - In [13] we have proved the existence of foliations of sufficiently high degree with prescribed linear holonomy group with respect to a given curve. Here we will consider the opposite situation when the degree of the curve is comparable to the degree of the foliation. More precisely we will consider foliations of degree $d \in \mathbb{N}$ which have an invariant smooth curve of degree $c \in \mathbb{N}$ such that $d \leq 2 c-2$ (remark that in all cases $c \leq d+1$ ). This inequality is equivalent to $Z(\mathscr{F}, C) \leq c^{2}$. As already pointed out it implies that the decomposition given by Lemma 2.1 is essentially unique.

Let us take a pair of transverse algebraic curves $C=\{F=0\}$ and $E$ defined by polynomials of degree $c \in \mathbb{N}$ and $k \in \mathbb{N}$ respectively; $C$ is supposed to be a smooth curve and $F$ a reduced polynomial. Denote by $F_{o l} l_{C, C \cap E}(d)$ the space of foliations of degree $d=c+k-2$ which leave $C$ invariant and have $C \cap E$ as the singular set along $C$. We define the Index Map $J(C, E)=I$ as

$$
\begin{aligned}
I: \mathbb{F o l}_{C, C \cap E}(d) & \rightarrow \mathscr{C}(C \cap E, \mathbb{C}) \\
\mathcal{F} & \mapsto(p \mapsto i(\mathcal{F}, C, p))
\end{aligned}
$$

where $\mathscr{G}(C \cap E, \mathbb{C})$ is the space of maps from $\Gamma$ to $\mathbb{C}$ and $i(\mathcal{F}, C, p)$ is in the index of $\mathscr{F}$ with respect to $C$ at the point $p$ (cf. [3]).

According to Lemma 2.1 , a foliation $\mathcal{F} \in \operatorname{Fol}_{C, C \cap E}(d)$ is defined by a 1 -form $\omega=G d F-(c / k) F d G+F \eta=0$; we may assume that $E=\{G=0\}$. A simple computation shows that

$$
\begin{equation*}
i(\mathcal{F}, C, p)=\frac{c}{k}-\operatorname{Res}\left(\left(\frac{\eta}{G}\right)_{\mid C}, p\right) \tag{2}
\end{equation*}
$$

where $\left(\frac{\eta}{G}\right)_{\mid C}$ means $i^{*}\left(\frac{\eta}{G}\right)$ for the inclusion $i: C \rightarrow \mathbb{P}^{2}$.

When $C$ and $E$ are transverse to each other at $p \in C \cap E$, we have

$$
\begin{equation*}
i(\mathcal{F}, C, p)=\frac{c}{k} \Leftrightarrow i^{*} \eta(p)=0 \tag{3}
\end{equation*}
$$

The equality $i^{*} \eta(p)=0$ means that $\mathcal{F}_{\eta}$ is tangent to $C$ at $p$.
If the foliation is the pencil $\mathscr{F}_{\Gamma}: G d F-(c / k) F d G=0$, all $k . c$ indexes at the points of $C \cap E$ are equal to $c / k$; a natural question to ask is whether the converse is true. This is not always the case (see [14] for a counterexample). Before stating the main result of this Section, we need a lemma; set $S_{l}=\mathrm{H}^{0}\left(\mathbb{P}^{2}, \Theta(l)\right)$ and $\left.\mathbb{S}_{l}=\mathbb{P H}^{0}\left(\mathbb{P}^{2}, \Theta(l)\right)\right)$ for $l>0$.

Lemma 2.2. - Let $c \geq k$. There exists a Zariski open subset $\mathcal{U}_{0}(c, k) \subset \mathbb{S}_{c} \times \mathbb{S}_{k}$ such that if $(C, E) \in \mathcal{U}_{0}(c, k)$ then $C$ and $E$ are transverse to each other and no foliation of degree $k-2$ is tangent to $C$ at the points of $C \cap E$.
Proof. - Let $X_{h}(n)$ be the set of homogeneous vector fields of $\mathbb{C}^{3}$ of degree $n$, and $H$ the set

$$
\left\{(F, G) \in S_{c} \times S_{k} ; \exists(Z, A, B) \in X_{h}(k-2) \times S_{k-3} \times S_{c-3} ; d F(Z)=A . F+B . G\right\}
$$

Then $H$ is an algebraic subvariety of $S_{c} \times S_{k}$. Let us show that $H$ is a strict subvariety. For that we take $F_{0} \in S_{c}$ as the equation of a plane rational curve of degree $c$ with nodal singularities and $G_{0}$ defining a plane curve of degree $k$ which is transverse to $\left\{F_{0}=0\right\}$. We know from the genus formula that $\left\{F_{0}=0\right\}$ has $\frac{(c-1) \cdot(c-2)}{2}$ nodal singularities. If $\left(F_{0}, G_{0}\right) \in H$, one has $D f_{0}\left(Z_{0}\right)=A_{0} \cdot F_{0}+B_{0} . G_{0}$ for a $\left(Z_{0}, A_{0}, B_{0}\right) \in$ $X_{h}(k-2) \times S_{c-3} \times S_{k-3}$. Let us compute the number of intersection points between $\left\{d F_{0}\left(Z_{0}\right)=0\right\}$ and $\left\{F_{0}=0\right\}$ :

- k.c points of $\left\{F_{0}=0\right\} \cap\left\{G_{0}=0\right\}$, which are smooth points of $\left\{F_{0}=0\right\}$.
- $(c-1)(c-2)$ points corresponding to the nodal singularities of $\left\{F_{0}=0\right\}$

We have then $k \cdot c+(c-1) \cdot(c-2)=(k+c-3) \cdot c$, contradiction.
Let now $\mathcal{U}(c, k)$ be the open subset of $S_{c} \times S_{k}$ of pairs of curves $(C, E)$ such that $C$ and $E$ are transverse to each other; finally we set $\mathcal{U}_{0}(c, k)=\mathcal{U}(c, k) \cap\left(S_{c} \times S_{k} \backslash H\right)$. Consider $(\bar{C}, \bar{E})=(\{\bar{F}=0\},\{\bar{G}=0\}) \in \mathcal{U}_{0}(c, k)$ and suppose that $d \bar{F}(\bar{Z})(p)=0$ at all points in $\bar{C} \cap \bar{E}$ for some $\bar{Z} \in X_{h}(k-2)$. By Noether's Theorem $([\mathbf{6}]), d \bar{F}(\bar{Z})=$ $\bar{A} . \bar{F}+\bar{B} . \bar{G}$ for some $(\bar{A}, \bar{B}) \in S_{c-3} \times S_{k-3}$, so that $(\bar{C}, \bar{E}) \in H$, contradiction unless $\bar{Z}=0$.

Remark. - The argument above is inspired in Severi's idea to prove the BrillNoether Theorem ([8], p. 240-244).

We have as a consequence:
Theorem 2. - Let $c \geq k$. There exists a Zariski open subset $\mathcal{U}_{1}(c, k) \subset \mathbb{S}_{c} \times \mathbb{S}_{k}$ such that if $(C, E) \in U_{1}(c, k)$ then $C$ is smooth, $C$ and $E$ are transverse to each other and $\jmath(C, E)$ is injective.

