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THE q-ANALOGUE OF THE WILD FUNDAMENTAL GROUP (II)

by

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Abstract. — In a previous paper, we defined q-analogues of alien derivations and stated their basic properties. In this paper, we prove the density theorem and the freeness theorem announced there.

***Résumé* (Le q-analogue du groupe fondamental sauvage (II)).** — Dans un article précédent nous avons défini les q-analogues des dérivations étrangères et leurs propriétés de base. Dans cet article nous démontrons le théorème de densité et d'indépendance que nous y avions annoncé.

1. Introduction

1.1. The problem. — In this paper we shall continue the study of the local meromorphic classification of q-difference modules. In [**10**] we gave such a classification in Birkhoff style, using normal forms and index theorems; this classification is complete in the "integral slope case". (One could extend it to the general case using some results of [**3**].)

In [**6**] we introduced a new approach of the classification, using a "fundamental group" and its finite dimensional representations, in the style of the Riemann-Hilbert correspondence for linear differential equations. At some abstract level, such a classification is well known: the fundamental group is the tannakian Galois group of the tannakian category of local meromorphic q-modules. But we wanted more information: our essential aim was to get a *smaller* fundamental group which is Zariski dense in the tannakian Galois group and to describe it *explicitly*, in the spirit of what was done by the first author for the differential case [**5**].

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In [6] we built a family of elements of the Lie algebra of the tannakian group, the q-alien derivations, we achieved our program for the one-level case and we announced the main results in the general case. The aim of the present paper is to give some proofs omitted in [6] for the multi-level case. We will finally give a more precise algebraic formulation of our results in [7], which will end the series.

1.2. Contents of the paper. — General notations and conventions are explained in the next paragraph 1.3. In section 2, we recall basic properties of the category $\mathcal{E}_1^{(0)}$ of linear analytic q-difference equations with integral slopes, and the structure and action of its Galois group $G_1^{(0)}$. In section 3, we recall the unipotent structure of the Stokes subgroup $\mathfrak{St}$ of $G_1^{(0)}$, and the construction (taken from [6]) of elements of the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$, the so-called *q-alien derivations*. Our "q-analogue of the wild fundamental group" is the Lie subalgebra they generate. We then prove in 3.2 and 3.3 our main results: density and a freeness property of the q-alien derivations. In section 4, we summarize what remains to be solved, and will be the contents of [7].

The paper is written so as to be read widely independently of [6] - granted the reader is willing to take on faith some key points. The principle of the proofs is almost purely tannakian, but we have stated explicitly the underlying methods and prerequisites. Moreover, they are described in a concrete, computational form (with a systematic use of matrices). Since neither q-difference equations, nor even tannakian methods are so popular, this may help the reader to get familiarized with either domain. Note that, since we heavily rely on transcendental tools, the methods here are, to a large extent, independent of those of M. van der Put and his coauthors.

1.3. General notations. — The notations are the same as in [6]. Here are the most useful ones.

We let $q \in \mathbf{C}$ be a complex number with modulus $|q| > 1$. We write σ_q the q-dilatation operator, so that, for any map f on an adequate domain in $\mathbf{C}$, one has: $\sigma_q f(z) = f(qz)$. Thus, σ_q defines a ring automorphism in each of the following rings: $\mathbf{C}\{z\}$ (convergent power series), $\mathbf{C}[[z]]$ (formal power series), $\mathcal{O}(\mathbf{C}^*)$ (holomorphic functions over $\mathbf{C}^*$), $\mathcal{O}(\mathbf{C}^*, 0)$ (germs at 0 of elements of $\mathcal{O}(\mathbf{C}^*)$). Likewise, σ_q defines a field automorphism in each of their fields of fractions: $\mathbf{C}(\{z\})$ (convergent Laurent series), $\mathbf{C}((z))$ (formal Laurent series over), $\mathcal{M}(\mathbf{C}^*)$ (meromorphic functions over $\mathbf{C}^*$), $\mathcal{M}(\mathbf{C}^*, 0)$ (germs at 0 of elements of $\mathcal{M}(\mathbf{C}^*)$)

The σ_q-invariants elements of $\mathcal{M}(\mathbf{C}^*)$ can be considered as meromorphic functions on the quotient Riemann surface $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. Through the mapping $x \mapsto z = e^{2\mathrm{i}\pi x}$, the latter is identified to the complex torus $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$, where $q = e^{2\mathrm{i}\pi\tau}$. Accordingly, we shall identify the fields $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ and $\mathcal{M}(\mathbf{E}_q)$. We shall write $a \mapsto \overline{a}$ the canonical projection map $\pi : \mathbf{C}^* \to \mathbf{E}_q$ and $[c;q] = \pi^{-1}(\overline{c}) = cq^{\mathbf{Z}}$ (a discrete logarithmic q-spiral).

Last, we shall have use for the function $\theta \in \mathcal{O}(\mathbf{C}^*)$, a Jacobi Theta function such that $\sigma_q \theta = z\theta$ and θ has simple zeroes along $[-1;q]$. One then puts $\theta_c(z) = \theta(z/c)$, so that $\theta_c \in \mathcal{O}(\mathbf{C}^*)$ satisfies $\sigma_q \theta_c = (z/c)\theta_c$ and θ_c has simple zeroes along $[-c;q]$.

2. Linear analytic q-difference equations

A linear analytic (resp. formal) q-difference equation (implicitly: at $0 \in \mathbf{C}$) is an equation:

$$\sigma_q X = AX, \tag{1}$$

where $A \in GL_n(\mathbf{C}(\{z\}))$ (resp. $A \in GL_n(\mathbf{C}((z)))$). There is an intrinsic description as a "q-difference module M_A", which runs as follows. We consider the operator Φ_A on $\mathbf{C}(\{z\})^n$ which maps a column vector X to $A^{-1}\sigma_q X$. This can be abstracted as a finite dimensional $\mathbf{C}(\{z\})$-vector space V endowed with a so-called "σ_q-linear automorphism" Φ. A q-difference module is such a pair $M = (V, \Phi)$. Here, we have $M_A = (\mathbf{C}(\{z\})^n, \Phi_A)$.

We shall here stick to the matrix model and, for all practical purposes, the reader may identify the equation (1), the matrix A and the q-difference module M_A with each other. For instance, we call *solution of A, or of (1), or of M_A* in some extension K of $\mathbf{C}(\{z\})$, on which σ_q operates, a column vector $X \in K^n$ such that $\sigma_q X = AX$. The *underlying space* of $A \in GL_n(\mathbf{C}(\{z\}))$ is $\mathbf{C}(\{z\})^n$.

2.1. Description of the tannakian structure. — We now proceed to describe the *tannakian category of analytic q-difference equations* $\mathcal{E}^{(0)}$. There is a similar description for the corresponding formal category $\hat{\mathcal{E}}^{(0)}$. The objects of $\mathcal{E}^{(0)}$ are linear analytic q-difference equations (1). A morphism from $A \in GL_n(\mathbf{C}(\{z\}))$ to $B \in GL_p(\mathbf{C}(\{z\}))$ is a matrix $F \in M_{p,n}(\mathbf{C}(\{z\}))$ such that:

$$(\sigma_q F)A = BF. \tag{2}$$

This just means that F sends any solution X of A to a solution FX of B. One can check that $\mathcal{E}^{(0)}$ is an abelian category. Indeed, it is the category of finite length left modules over the euclidean non-commutative ring $\mathcal{D}_{q,K}$ of q-difference operators over $K = \mathbf{C}(\{z\})$.

The abelian category $\mathcal{E}^{(0)}$ is endowed with a tensor structure. The tensor product $A_1 \otimes A_2$ of two objects (resp. the tensor product $F_1 \otimes F_2$ of two morphisms) is just the Kronecker product of the matrices; of course, we must define a consistent way of identifying $\mathbf{C}^n \otimes \mathbf{C}^p$ with $\mathbf{C}^{np}$, or $\mathbf{C}(\{z\})^n \otimes \mathbf{C}(\{z\})^p$ with $\mathbf{C}(\{z\})^{np}$ (see, for instance **[11]**).

The unit object $\underline{1}$ (which is neutral for the tensor product) is the matrix $(1) \in GL_1(\mathbf{C}(\{z\})) = \mathbf{C}(\{z\})^*$, with underlying space $\mathbf{C}(\{z\})$. The object $\underline{1}$ of course corresponds to the "trivial" equation [(1)] $\sigma x = x$. One easily checks that the space $\mathrm{Hom}(\underline{1}, A)$ of morphisms from $\underline{1}$ to A is exactly the space of solutions of A in $\mathbf{C}(\{z\})$, or, equivalently, the space of fixed points of Φ_A in $\mathbf{C}(\{z\})^n$. We shall write $\Gamma(A)$ or $\Gamma(M_A)$

(1) In differential Galois theory, the matrix A of a system is in $M_n(\mathbf{C}(\{z\}))$ (rather than in GL_n), the trivial equation is $x' = 0$, etc. The theory of q-difference equations rather has a multiplicative character

that space, as it is similar to a space of global sections (and, indeed, can be realised as such, see [**14**]).

The characterization (2) of morphisms can itself be seen as a q-difference equation $\sigma_q F = BFA^{-1}$. This means that there is an "internal Hom" object, which can be described in the following way. Consider the linear operator $F \mapsto BFA^{-1}$ on the vector space $M_{p,n}(\mathbf{C}(\{z\}))$. Through identification of $M_{p,n}(\mathbf{C}(\{z\}))$ with $\mathbf{C}(\{z\})^{np}$, this operator is described by a matrix in $GL_{np}(\mathbf{C}(\{z\}))$, which yields the desired equation. We shall write $\underline{\mathrm{Hom}}(A,B)$ the corresponding object. Thus, one gets:

$$\Gamma(\underline{\mathrm{Hom}}(A,B)) \simeq \mathrm{Hom}(\underline{1}, \underline{\mathrm{Hom}}(A,B)) \simeq \mathrm{Hom}(A,B). \tag{3}$$

Actually, this is a special case of the following canonical isomorphism::

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B,C)) \simeq \mathrm{Hom}(A \otimes B, C). \tag{4}$$

The reader will check that the object $\underline{\mathrm{Hom}}(A,\underline{1})$ has the following description. The underlying space is $M_{1,n}(\mathbf{C}(\{z\}))$, which we identify with $\mathbf{C}(\{z\})^n$. The corresponding matrix for the linear operator $F \mapsto FA^{-1}$ is the *contragredient matrix* $A^\vee = {}^tA^{-1}$. We call the object $A^\vee$ the *dual* of the object A. From this, we get yet another construction of the internal Hom:

$$\underline{\mathrm{Hom}}(A,B) \simeq A^\vee \otimes B. \tag{5}$$

We summarize these properties by saying that $\mathcal{E}^{(0)}$ is a tannakian category. This is halfway to showing that it is (isomorphic to) the category of representations of a proalgebraic group, our hoped for Galois group. To get further, one needs a *fiber functor on* $\mathcal{E}^{(0)}$. This was defined and, to some extent, studied in full generality in [**13**], [**12**] and [**6**]. However, for our strongest results, we need to restrict to the case of *integral slopes*.

2.2. Equations with integral slopes. — In [**13**], one defined the Newton polygon of a q-difference module (analytic or formal). This consists in slopes [(2)] $\mu_1 < \cdots < \mu_k$ (rational numbers) together with ranks (or multipicities) $r_1, \ldots, r_k$ (positive integers). We shall say that a module is *pure isoclinic* if it has only one slope and that it is *pure* [(3)] if it is a direct sum of pure isoclinic modules. We call *fuchsian* a pure isoclinic module with slope 0. The Galois theory of fuchsian modules was studied in [**11**]. Pure modules are irregular objects without *wild monodromy*, as follows from [**10**], [**12**] and [**6**].

The tannakian subcategory of $\mathcal{E}^{(0)}$ made up of pure modules is called $\mathcal{E}_p^{(0)}$. Modules with integral slopes also form tannakian subcategories, which we write $\mathcal{E}_1^{(0)}$ and $\mathcal{E}_{p,1}^{(0)}$. *From now on, we restrict to the case of integral slopes*. Our category of interest is therefore $\mathcal{E}_1^{(0)}$ and we shall now start its description.

(2) Note that here, as in [**6**], we have changed the definitions of slopes. Those used here are the opposites of those used in [**13**], [**8**]and [**12**].

(3) Here again, starting with [**6**], we changed our terminology: we now call pure isoclinic (resp. pure) what was previously called pure (resp. tamely irregular). The latter are called *split modules* in [**3**].

Any equation in $\mathscr{E}_1^{(0)}$ can be written in the following *standard form*:

$$A = \begin{pmatrix} z^{\mu_1} A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & U_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{\mu_k} A_k \end{pmatrix}, \tag{6}$$

where the slopes $\mu_1 < \cdots < \mu_k$ are integers, $r_i \in \mathbf{N}^*$, $A_i \in GL_{r_i}(\mathbf{C})$ $(i = 1, \ldots, k)$ (those μ_i and r_i make up the Newton polygon of A) and:

$$\forall i, j \text{ s.t. } 1 \leq i < j \leq k \ , \ U_{i,j} \in \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\})).$$

We actually can, and will, require the blocks $U_{i,j}$ to have all their coefficients in $\mathbf{C}[z, z^{-1}]$. Then any morphism $F : A \to B$ between two matrices in standard form is easily seen to be meromorphic at 0 (by definition) and holomorphic all over $\mathbf{C}^*$; this is because the equation $\sigma_q F = BFA^{-1}$ allows one to propagate the regularity near 0 to increasing neighborhoods.

We moreover say that A is in *polynomial* standard form if each block $U_{i,j}$ with $1 \leq i < j \leq k$ has coefficients in $\sum\limits_{\mu_i \leq d < \mu_j} \mathbf{C} z^d$. It was proved in [**10**] that any object in $\mathscr{E}_1^{(0)}$ is analytically equivalent to one written in polynomial standard form (in essence, this is due to Birkhoff and Guenther). Last, we say that A is in *normalized* standard form is if all the eigenvalues of all the blocks A_i are in the fundamental annulus $\{z \in \mathbf{C}^* \mid 1 \leq |z| < |q|\}$. Any standard form can be normalized through shearing transformations. Note that polynomial standard form is stable under tensor product, while normalized standard form is not.

The standard form (6) above expresses the existence of a *filtration by the slopes* ([**13**]). The functoriality of the filtration moreover entails that a morphism $F : A \to B$ is also upper triangular (by blocks) in the following sense: if the slopes of $B \in GL_p(\mathbf{C}(\{z\}))$ are $\nu_1 < \cdots < \nu_l$, with ranks $s_1 < \cdots < s_l$, then the morphism $F \in M_{p,n}(\mathbf{C}(\{z\}))$ from A to B has only non null blocks $F_{i,j} \in M_{s_j, r_i}(\mathbf{C}(\{z\}))$, $1 \leq i \leq k$, $1 \leq j \leq l$ for $\nu_j \leq \mu_i$.

To the matrix A and module $M = M_A$ is associated the graded module $\mathrm{gr} M = M_0 = M_{A_0}$ with block diagonal matrix:

$$A_0 = \begin{pmatrix} z^{\mu_1} A_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & z^{\mu_k} A_k \end{pmatrix}, \tag{7}$$

The graded module M_0 is the direct sum $P_1 \oplus \cdots \oplus P_k$, where each module P_i is pure of rank r_i and slope μ_i and corresponds to the matrix $z^{\mu_i} A_i$. The functor $M \rightsquigarrow \mathrm{gr} M$