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# TRAVAUX DE ZINK

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## 1. INTRODUCTION AND PRELIMINARY DEFINITIONS AND RESULTS

Fix a prime number $p$. All rings considered will be $\mathbf{Z}_{(p)}$-algebras. If $R$ is a ring we will consider $p$-divisible groups over $R$ and in particular those which are formal groups. If $\frac{1}{p} \in R$, then $p$-divisible groups are étale and consequently given by continuous representations $\rho : \pi_1(\mathrm{Spec}(R)) \to \mathrm{GL}_h(\mathbf{Z}_p)$. Hence we shall assume $p$ is either nilpotent in $R$ or $R$ is separated and complete for a topology having a neighborhood basis of 0 consisting of ideals and that $p$ is topologically nilpotent.

With these conventions, the aim of the various Dieudonné theories is to classify the category of $p$-divisible groups over $R$ via functors to categories living in the realm of (semi)linear algebra. One should think of them as analogous to the functor $G \mapsto \mathrm{Lie}(G)$ which establishes an equivalence of categories between formal groups and Lie algebras when $R$ is a $\mathbf{Q}$-algebra. We will not give an overview of the various Dieudonné theories, but rather concentrate on the most recent, Zink's theory of displays. Nevertheless it will be necessary for us to relate Zink's theory to Cartier's theory and to the crystalline theory. We refer the reader to [Ta], [Ser], [Gr1], [Gr2], [Dem], [Fon1] for $p$-divisible groups, to [Car1], [Car2], [Haz], [Laz], [Z1], [Z2] for Cartier theory, to [Gr1], [Gr2], [MM], [M], [BBM], [BM1], [BM2], [dJ2], [dJM] for crystalline Dieudonné theory.

If $R$ is a perfect field of characteristic $p$, these theories are, for $p$-divisible groups (formal in the case of Cartier's theory), all equivalent. Indeed it was one of Zink's motivations in developing his theory to relate the Cartier theory to the crystalline theory. But, in establishing properties of his theory, he uses both the Cartier and the crystalline theories. Hence there is a symbiotic relationship between the three theories.

We refer to [Bour] for the standard facts about the Witt vector ring, $W(R)$. We write $w_n : W(R) \to R$ for the ghost component maps, $f : W(R) \to W(R)$ for the Frobenius ring endomorphism and $v : W(R) \to W(R)$ for the additive Verschiebung endomorphism. Let $I_R = \mathrm{Ker}(w_0) = \mathrm{im}(v)$. If $a \in R$, $[a]$ denotes its Teichmüller representative.

LEMMA 1.1. — *If $R$ is separated and complete in the $p$-adic topology, then $W(R)$ is separated and complete in both its $p$-adic and $I_R$-adic topologies. If $p$ is nilpotent in $R$, these topologies coincide and it is finer than the $v$-adic topology.*

DEFINITION 1.2. — *A display $\mathscr{P}$ over $R$ is a quadruple $(P, Q, F, F_1)$ where $P$ is a finitely generated projective $W(R)$-module, $Q$ a submodule, $F : P \to P$, $F_1 : Q \to P$ are $f$-semilinear such that*

(i) $I_R P \subset Q$.

(ii) $0 \to Q/I_R P \to P/I_R P \to P/Q \to 0$ is a split sequence of $R$-modules.

(iii) $P$ is generated by $\mathrm{im}(F_1)$.

(iv) $F_1(v(\xi)x) = \xi F(x)$ for $\xi \in W(R)$, $x \in P$.

If $u : M \to N$ is a $f$-semilinear map of $W(R)$-modules, we set $M^{(1)} = W(R) \otimes_{f,W(R)} M$ for the extension of scalars using $f$ and denote by $u^\sharp : M^{(1)} \to N$ the associated linear map.

With the obvious notion of morphisms, displays form an additive category and, if we define a morphism of displays $u : \mathscr{P}' \to \mathscr{P}$ to be an admissible monomorphism (resp. epimorphism) provided $u : P' \to P$ is injective (resp. surjective) and $u^{-1}(Q) = Q'$ (resp. $u(Q') = Q$), we equip $\mathrm{Displays}_R$ with the structure of an exact category.

DEFINITION 1.3. — *A normal decomposition for a display $\mathscr{P}$ over $R$ is a direct sum decomposition $P = L \oplus T$ such that $Q = L \oplus I_R T$.*

If $R$ is a $p$-adic ring, in particular if $p$ is nilpotent in $R$, normal decompositions always exist. This is a consequence of the fact that finitely generated projective modules can always be lifted for surjections $A \to B$ whose kernel is a nilideal or such that $A$ is separated and complete for the topology given by powers of the kernel.

*Examples.–* (i) The display corresponding to the formal multiplicative group $\mathscr{G} = (W(R), I_R, f, v^{-1})$.

(ii) If $R = k$, a perfect field of characteristic $p$, $M \mapsto \mathscr{P}_M = (M, V(M), F, V^{-1})$ establishes an equivalence of categories between Dieudonné modules over $k$ and displays over $k$.

From now on we assume $p$ is nilpotent in $R$, unless we explicitly state the contrary.

If $u : R \to R'$ is a ring homomorphism and $\mathscr{P}$ is a display over $R$, the base changed display $u_*(\mathscr{P})$ is the display over $R'$, $\mathscr{P}' = (P', Q', F', F_1')$, where $P' = W(R') \otimes_{W(R)} P$, $Q' = \mathrm{Ker}(P' \to R' \otimes_R P/Q)$, $F' = f \otimes F$ and $F_1'$ is determined by

$$F_1'\big(v(\xi) \otimes x\big) = \xi \otimes F(x)\,,\ \xi \in W(R')\,,\ x \in P$$

and

$$F_1'\big(\xi \otimes y\big) = f(\xi) \otimes F_1(y)\,,\ \xi \in W(R')\,,\ y \in Q.$$

Using a normal decomposition, it is easy to show that $F_1'$ exists and $\mathscr{P}'$ is a display.

Definition 1.4. — *Let $\mathscr{P}, \mathscr{P}'$ be displays over $R$. A bilinear form of displays $(\ ,\ ) : \mathscr{P} \times \mathscr{P}' \to \mathscr{G}$ is a bilinear map $P \times P' \to W(R)$ such that $v(F_1 y, F_1' y') = (y, y')$ for $y \in Q$, $y' \in Q'$.*

If $\mathscr{P}$ is a display over $R$, its dual display $\mathscr{P}^t = (P^\vee, \widehat{Q}, F, F_1)$ where $P^\vee = \mathrm{Hom}_{W(R)}(P, W(R))$, $\widehat{Q} = \{z \in P^\vee | z(Q) \subset I_R\}$ and $F$ and $F_1$ are determined by

$$\begin{array}{rcll} (F_1 x, Fz) & = & f(x,z) & \text{for } x \in Q\,,\ z \in P^\vee \\ (Fx, Fz) & = & pf(x,z) & \text{for } x \in P\,,\ z \in P^\vee \\ (Fx, F_1 z) & = & f(x,z) & \text{for } x \in P\,,\ z \in \widehat{Q} \\ v(F_1 x, F_1 z) & = & (x,z) & \text{for } x \in Q\,,\ z \in \widehat{Q}. \end{array}$$

We have a canonical isomorphism

$$\mathrm{Bil}(\mathscr{P}, \mathscr{P}'; \mathscr{G}) \simeq \mathrm{Hom}(\mathscr{P}', \mathscr{P}^t).$$

Proposition 1.5. — *There is a unique linear map $V^\sharp : P \to P^{(1)}$ determined by $V^\sharp(\xi F x) = p\xi \otimes x$, $V^\sharp(\xi F_1 y) = \xi \otimes y$, for $\xi \in W(R)$, $x \in P$, $y \in Q$.*

This is established by taking a normal decomposition $P = L \oplus T$, showing that $F_1^\sharp \oplus F^\sharp : L^{(1)} \oplus T^{(1)} \to P$ is bijective and defining $V^\sharp$ to be the composite

$$(\mathrm{id} \oplus p \cdot \mathrm{id}) \circ (F_1^\sharp \oplus F^\sharp)^{-1} : P \to L^{(1)} \oplus T^{(1)} = P^{(1)}.$$

One has $F^\sharp \circ V^\sharp = p \cdot \mathrm{id}_P$, $V^\sharp \circ F^\sharp = p \cdot \mathrm{id}_{P^{(1)}}$. If $P^{(i)}$ is the scalar extension of $P$ using $f^i$, then $V^\sharp$ gives rise to $V_i^\sharp : P^{(i)} \to P^{(i+1)}$.

Definition 1.6. — *$\mathscr{P}$ satisfies the nilpotence condition or $\mathscr{P}$ is a* nilpotent display *provided there is an $N$ such that $V_N^\sharp \circ V_{N-1}^\sharp \circ \cdots \circ V^\sharp$ is zero modulo $I_R + p\,W(R)$.*

*Remark 1.7.* — In [Z5], displays were called $3n$-displays ($3n$ for "not necessarily nilpotent") and nilpotent displays were called displays. We follow Zink's more recent terminology (*cf.* his Paris 13 lectures of February, 2006) here. Also in [Z5], $F_1$ was denoted by $V^{-1}$. Zink and Langer have initiated a theory of higher displays, [LZ2], in which $P = P_0$, $Q = P_1$ and there are higher $P_i$ and $F_i : P_i \to P$. For this reason we write, following Zink, $F_1$ instead of his original $V^{-1}$.

*Remark 1.8.* — Locally on Spec($R$), if $L \oplus T$ is a normal decomposition we will have $L$ and $T$ free modules and if $T$ has basis $\{e_1, \ldots, e_d\}$ and $L$ has basis $\{e_{d+1}, \ldots, e_h\}$, the map $F_1^\sharp \oplus F^\sharp$ will be expressed in terms of these bases by a matrix $(\alpha_{ij}) \in \mathrm{GL}_h(W(R))$. Conversely any such invertible matrix will determine a display. If the matrix $(\alpha_{ij})$ has inverse $(\beta_{k\ell})$, and $B$ is the $(h-d) \times (h-d)$ matrix with entries in $R/p\,R$ given by $B = (w_0(\beta_{k\ell}) \bmod p)_{k,\ell=d+1,\ldots,h}$, then $\mathscr{P}$ is nilpotent if and only if there is an $N$ such that

$$B^{(p^N)} \ldots B = 0,$$

where $B^{(p^i)}$ is the matrix obtained by applying the $i$-th iterate of Frobenius to $B$.

If $e_i$ is a basis for a free module over the Cartier ring, then the relations

$$\begin{aligned} Fe_i &= \sum \alpha_{ji} e_j\,, & i &= 1, \ldots d\,; \\ e_i &= V\Big(\sum \alpha_{ji} e_j\Big)\,, & i &= d+1, \ldots h \end{aligned}$$

define a reduced Cartier module. Relations of this form were called by Norman [N] "displayed structural equations" of a reduced Cartier module. This is the origin of Zink's use of the term display.

*Remark 1.9.* — Let $S \overset{u}{\twoheadrightarrow} R$ be a surjection whose kernel is a nilideal. Let $\mathscr{P}$ be a display over $R$. Then there is a display $\mathscr{P}'$ over $S$ and an isomorphism $u_*(\mathscr{P}') \xrightarrow{\sim} \mathscr{P}$.

This is proven using the fact that finitely generated projective $W(R)$-modules can be lifted to finitely generated projective $W(S)$-modules and using normal decompositions. Nakayama's lemma then shows that lifting modules are determined up to isomorphism (non-unique!).

If $\mathscr{P}/R$ is a nilpotent display and $\mathscr{P}'$ is a lifting to $S$, then $\mathscr{P}'$ is nilpotent too. This is clear as $\mathrm{Ker}(S \to R)$ is a nilideal.

We ask about the ambiguity in the *lifting* $\mathscr{P}'$ of $\mathscr{P}$. If $\mathscr{P}' = (P', Q', F', F_1')$, $J = \mathrm{Ker}(S \overset{u}{\twoheadrightarrow} R)$ and $\alpha : P' \to W(J) \otimes_{W(S)} P'$, we define a display $\mathscr{P}'_\alpha$ over $S$ lifting $\mathscr{P}$ by $\mathscr{P}'_\alpha = (P', Q', F'_\alpha, F'_{1\alpha})$, where $F'_\alpha(x) = F'x - \alpha(F'x)$, for $x \in P'$, $F'_{1\alpha}(y) = F'_1 y - \alpha(F'_1 y)$, for $y \in Q'$. Then $\mathscr{P}'_\alpha$ is a display and Zink shows any lifting of $\mathscr{P}$ is isomorphic to a $\mathscr{P}'_\alpha$.

*Remark 1.10.* — Assume $p \cdot 1_R = 0$. Let $\mathscr{P}$ be a display over $R$, $\mathscr{P}^{(p)}$ be the display over $R$ given by $(\mathrm{Frob})_*\mathscr{P}$. Then $V^\sharp$ commutes with $F$ and $F_1$ and hence defines a morphism of display $\mathscr{F}_{\mathscr{P}} : \mathscr{P} \to \mathscr{P}^{(p)}$. Similarly $F^\sharp$ defines a morphism of displays $\mathscr{V}_{\mathscr{P}} : \mathscr{P}^{(p)} \to \mathscr{P}$. Of course both composites are multiplications by $p$.

If $R \to R'$ is a ring homomorphism, there is an obvious notion of a descent datum for $\mathscr{P}'$ a $R'$-display and, if $\mathscr{P}$ is a $R$-display, $\mathscr{P}_{R'}$ has a canonical descent datum, can.

Zink proves:

PROPOSITION 1.11. — *If $R \to R'$ is faithfully flat and $p$ is nilpotent in $R$, then $\mathscr{P} \mapsto (\mathscr{P}_{R'}, \mathrm{can})$ is an equivalence of categories between* Displays$/R$ *and the category of $R'$-displays equipped with descent data. The same is true for nilpotent displays.*

## 2. THE CRYSTALS ASSOCIATED TO DISPLAYS

We refer to [Ber] for a detailed discussion of crystals, crystalline cohomology,... and recall the bare minimum here. An ideal $J \subset A$ has divided powers if we are given maps $\gamma_n : J \to J$, $n \geq 1$, satisfying axioms imposed by thinking of $\gamma_n(x)$ as $\frac{x^n}{n!}$. The ideal $(p) \subset \mathbf{Z}_{(p)}$ has unique divided powers since $\frac{p^n}{n!} \in (p)$. It follows that for any ring $A$, $p \cdot A$ has divided powers. If $J \subset A$ is an ideal with divided powers we require that its divided powers agree with those on $J \cap pA$. This is called the compatibility condition. If $R$ is a $\mathbf{Z}_{(p)}$-algebra, then $I_R \subset W(R)$ has canonical divided powers which are compatible with those on $p \cdot W(R)$. These are determined by $\gamma_n\big(v(x)\big) = \frac{p^{n-1}}{n!} v(x^n)$, [Gr2]. The ideals $v^m(W(R))$ are sub-divided power ideals. We refer to [Ber] for the definition of nilpotent divided powers and to [M], [Z3] for a weaker notion.

We continue to assume $p$ is nilpotent in $R$. If $A$ is an $R$-algebra, a divided power thickening of $A$ is a surjection $A' \overset{\pi}{\twoheadrightarrow} A$ such that $p$ is nilpotent in $A'$ and $\mathrm{Ker}(\pi)$ is equipped with divided powers (satisfying the compatibility condition). A morphism of divided power thickenings is a commutative diagramm

$$(*) \qquad \begin{array}{ccc} A' & \xrightarrow{\pi} & A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ B' & \xrightarrow{\widetilde{\pi}} & B \end{array}$$

such that $\mathrm{Ker}(\pi)$, $\mathrm{Ker}(\widetilde{\pi})$ have divided powers, $\psi(\gamma_n(x)) = \gamma_n(\psi(x))$, $n \geq 1$ for $x \in \mathrm{Ker}\,\pi$.

A *crystal* in *modules* $M$ on $R$ is the giving for every divided power thickening $A' \overset{\pi}{\twoheadrightarrow} A$ of a $A'$-module, $M_{(A' \xrightarrow{\pi} A)}$, and for every morphism of divided power thickenings of an isomorphism

$$T_{(\psi,\phi)} : B' \underset{A'}{\otimes} \big(M_{(A' \xrightarrow{\pi} A)}\big) \xrightarrow{\sim} M_{(B' \xrightarrow{\widetilde{\pi}} B)},$$

these isomorphisms being required to satisfy the obvious transitivity condition.

Similarly we define a Witt-crystal on $R$ as the giving for any divided power thickening of an $R$-algebra $\big(A' \xrightarrow{\pi} A\big)$ of a $W(A')$-module $K_{(A' \xrightarrow{\pi} A)}$ together with, for