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THE WAVE MAP PROBLEM.
SMALL DATA CRITICAL REGULARITY
[after T. Tao]

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1. INTRODUCTION

The purpose of this paper is to describe the wave map problem

$$\Box\phi = -\phi\,(\partial_\alpha\phi\cdot\partial^\alpha\phi)\,, \qquad (1)$$
$$\phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1$$

where ϕ is a map $\phi : \mathbb{R}^{n+1} \to \mathbb{S}^{m-1} \subset \mathbb{R}^m$, and its analogs for other target manifolds, with a specific focus on the small data critical regularity results of T. Tao, contained in the following

Theorem 1 ([30], [31]). — *Let $n \geq 2$ and $s > \frac{n}{2}$. The solution of the Cauchy problem (1) with initial data $(\phi_0, \phi_1) \in (\mathbb{S}^{m-1}, T\mathbb{S}^{m-1})$ in $(\dot H^s, \dot H^{s-1})$ can be extended uniquely to a global solution $(\phi(t), \partial_t\phi(t)) \in (\dot H^s, \dot H^{s-1})$ on $\mathbb{R}^{n+1}$ provided that the initial data (ϕ_0, ϕ_1) has a sufficiently small $(\dot H^{\frac{n}{2}}, \dot H^{\frac{n}{2}-1})$ norm:*

$$\|\phi_0\|_{\dot H^{\frac{n}{2}}(\mathbb{R}^n)} + \|\phi_1\|_{\dot H^{\frac{n}{2}-1}(\mathbb{R}^n)} < \epsilon.$$

These results imply that in dimensions $n \geq 3$, despite the fact that the wave map problem is *supercritical* relative to a conserved energy and there exist solutions blowing up in finite time, its classical solutions with $\mathbb{S}^{m-1}$ target can be extended globally in time as long as the initial data has a small scale-invariant $\dot H^{\frac{n}{2}}$ norm(1). In the critical dimension $n = 2$ the result is particularly exciting as it implies that a solution exists globally as long as it has a small energy.

(1)Here and in what follows we will denote the initial data by $\phi[0] = (\phi_0, \phi_1)$ and will say that $\phi[0] \in H^s$ meaning $(\phi_0, \phi_1) \in H^s \times H^{s-1}$.

The problem (1) arises as an Euler-Lagrange equation corresponding (formally) to the critical points of the Lagrangian density:

$$\mathscr{L}[\phi] \ = \frac{1}{2}(\partial_\alpha\phi\cdot\partial_\beta\phi)\ m^{\alpha\beta}, \tag{2}$$

where $m_{\alpha\beta}$ is the Minkowski metric on $\mathbb{R}^{n+1}$. The density $\mathscr{L}[\phi]$ gives rise to the Minkowski analog of the harmonic map problem on $\mathbb{R}^n$, in which the energy density is given by $\frac{1}{2}\nabla\phi\cdot\nabla\phi$ and the critical points, harmonic maps $\phi:\mathbb{R}^n\to\mathbb{S}^{m-1}$, satisfy the equation

$$\Delta\phi = -\phi(\nabla\phi\cdot\nabla\phi).$$

The equation (1) belongs to the more general class of wave map problems, in which ϕ is a map from an $(n+1)$-dimensional Lorentzian manifold $(\mathscr{M}, g)$ to a Riemannian manifold $(\mathscr{N}, h)$. The map ϕ is a solution of the Euler-Lagrange equations:

$$D^\alpha\partial_\alpha\phi = 0, \tag{3}$$

corresponding to the Lagrangian density:

$$\mathscr{L}[\phi] \ = \ \frac{1}{2}\ h_{ij}\ (\partial_\alpha\phi^i\partial_\beta\phi^j)\ g^{\alpha\beta}. \tag{4}$$

Here $\{\phi^i\}$ denote local coordinates on $\mathscr{N}$. D is the pull-back of the Levi-Civita connection on $T\mathscr{N}$ to the bundle $\phi^*(T\mathscr{N})$. In terms of the local coordinates $\{\phi^i\}$ this pull-back connection acting on sections of $\phi^*(T\mathscr{N})$ reads:

$$D_\alpha \ = \ \nabla_\alpha + \overline{\Gamma}^k_{\alpha j}\ , \qquad\qquad \overline{\Gamma}^k_{\alpha j} \ = \ \Gamma^k_{ij}(\phi)\partial_\alpha\phi^i\ , \tag{5}$$

where Γ^k_{ij} is the Christoffel symbol in the coordinates $\{\phi^i\}$ and ∇ is a covariant derivative on $T\mathscr{M}$. The wave-map equation (3) has the form:

$$\Box_g\phi^k \ = \ -\Gamma^k_{ij}(\phi)\, g^{\alpha\beta}(\partial_\alpha\phi^i\partial_\beta\phi^j). \tag{6}$$

In particular in the case of a wave map problem from Minkowski space $(\mathbb{R}^{n+1}, m)$ the map ϕ verifies the equation

$$\Box\phi = -\Gamma(\phi)(\partial_\alpha\phi, \partial^\alpha\phi). \tag{7}$$

The wave map problem appears naturally in solid-state physics, theory of topological solitons, Quantum Field Theory and General Relativity:

Topological solitons. — One of the simplest non-trivial models with *topological soliton* solutions is the $(2+1)$ dimensional Lorentz invariant $O(3)$ classical σ-model which is nothing else but a $(2+1)$-dimensional wave map problem with $\mathbb{S}^2$ target. It arises in the study of a continuum limit of an isotropic anti-ferromagnet, [8]. Topological solitons in this model are the *static* solutions (harmonic maps from $\mathbb{R}^2\to\mathbb{S}^2$ of the equation

$$\Box\phi = -\phi(\partial_\alpha\phi\cdot\partial^\alpha\phi)$$

which minimize the energy (conserved under evolution)

$$E[\phi] = \int_{\mathbb{R}^2} \left(|\partial_t \phi|^2 + |\nabla_x \phi|^2\right) dx$$

in a given homotopy class. Such maps satisfy the Bogomol'nyi equation

$$\partial_i \phi = \pm \epsilon_{ij} \phi \times \partial_j \phi$$

and are thought to represent meta-stable particles, [1]. Here ϵ_{ij} is an anti-symmetric tensor in two dimensions. The important feature of this model, common to all $(2+1)$-dimensional wave map problems, is its *criticality.* Both the equation and the conserved energy $E[\phi]$ are invariant under scaling transformations $\phi(t,x) \to \phi(\lambda t, \lambda x)$. The problem displays a fascinating interplay between the infinite dimensional wave map dynamics defined by (1) and a finite dimensional dynamics generated by restricting the full dynamics to the moduli space of static solutions (e.g. self-shrinking $(\lambda \to 0)$ of harmonic maps), see e.g. [20], ultimately leading to the existence of large data solutions of (1) blowing up in finite time, [23].

General Relativity. — The wave map problem on a curved $(2+1)$-dimensional background with an $\mathbb{H}^2$ target arises in the $\mathbb{U}(1)$ symmetry reduction of the Einstein vacuum equations. In this case one starts with a $(\mathbf{M}, \mathbf{g})$ Lorentzian (3+1)-dimensional manifold with Ricci curvature

$$\mathbf{R}_{\alpha\beta} = 0.$$

Under the assumption that $(\mathbf{M}, \mathbf{g})$ is invariant under the group action of $U(1)$ which orbits are space-like the metric $\mathbf{g}$ can be decomposed

$$\mathbf{g} = e^{-2\gamma} g + e^{2\gamma} (\theta)^2$$

where g is a Lorentzian metric on a $(2+1)$-dimensional manifold $\mathscr{N} = (\Sigma \times \mathbb{R})$ and $\theta = dx^3 + A_a dx^a$ with $a = 0,1,2$ local coordinates on $\mathscr{N}$ and x^3 a coordinate along the orbit. The equations $\mathbf{R}_{a3} = 0$ (and the assumption of triviality of the first cohomology class of Σ) imply that

$$dA = \frac{1}{2} e^{-4\gamma} \star d\omega,$$

where $*$ is the Hodge dual relative to the metric g and a scalar function ω is called a twist potential. The equation $\mathbf{R}_{33} = 0$ implies that

$$\Box_g \gamma + \frac{1}{2} e^{-4\gamma} g^{ab} \partial_a \omega \, \partial_b \omega = 0,$$
$$\Box_g \omega - 4 g^{ab} \partial_a \omega \, \partial_b \gamma = 0$$

which can be recognized as a wave map equation from $(\Sigma \times \mathbb{R}, g)$ into the hyperbolic space $\mathbb{H}^2$ with the metric

$$2(d\gamma)^2 + \frac{1}{2} e^{-4\gamma} (d\omega)^2.$$

Note that the wave map evolves on a dynamic background with the metric g, which itself depends on the wave map. This coupling is determined by satisfying the remaining equations $\mathbf{R}_{ab} = 0$. The only result available in this fully nonlinear context is a small data global stability in the expanding direction statement for solutions with Σ a compact surface with genus greater than one, metric $g = -dt^2 + t^2\sigma$ with σ a metric of scalar curvature -1 on Σ and the wave map $\phi = 0$, see [6].

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2. SUMMARY OF QUESTIONS AND RESULTS FOR THE WAVE MAP PROBLEM FROM MINKOWSKI SPACE

Traditionally(2), as the wave map equation is a hyperbolic evolution problem, one is interested in the questions of local and global in time existence and uniqueness of solutions, existence of solutions blowing up in finite time and stability of static or other "preferred"(3) solutions. The wave map equation from Minkowski space is invariant under the scaling transformation $\phi(t, x) \to \phi(\lambda t, \lambda x)$, which also preserves the $\dot{H}^{\frac{n}{2}}$ Sobolev norm. On this basis and in view of a geometric nature of the problem, our experience suggests that we could expect(4) that:

Local in time solutions exist and unique for any initial data $\phi[0] \in H^s$ with $s > n/2$.

Solutions with data with a small $\dot{H}^{\frac{n}{2}}$-norm can be extended globally in time.

Large data classical solutions can be extended globally in time for the $(2+1)$-dimensional (*critical*) wave map problem, where the scale invariant space $\dot{H}^1$ coincides with a conserved energy space, at least in the case of a target manifold of negative curvature, in analogy with the harmonic map heat flow.

Large data classical solutions can be extended globally in time for the $(1+1)$-dimensional wave map problem, where the scale invariant space $\dot{H}^{\frac{1}{2}}$ is larger (subcritical) than the energy space.

(2) The connection of the wave problem to QFT and GR may present an additional set of questions.

(3) An example of such a solution is $\phi = \gamma(u)$ where γ is a geodesic on $(\mathcal{N}, h)$ and u verifies the wave equation $\Box u = 0$.

(4) Just on the basis of presented here "evidence" perhaps a more appropriate term here would be "hope" as in some other problems these expectations have not been yet fulfilled or simply turned out to be wrong. For the wave map problem these expectations are more grounded due to the referred to above geometric origin of the problem, which makes available various cancellation properties (e.g. the expression $\partial^\alpha\phi \cdot \partial_\alpha\phi$ is an example of a *null form* eliminating parallel interactions of free waves).

Below we briefly (and incompletely, sometimes referring to just the final result) summarize known results (a good survey of the wave map problem is given in [34]):

Existence and uniqueness of local in time solutions in H^s with $s > n/2$ is in [13], [15] and [11] in dimension $n = 1$.

Small data global existence in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}}$ in dimensions $n \geq 2$ is shown in [30], [31]. Extensions to other targets are in [14], [25], [21], [16], [17], [35].

Large data global existence for the $(1+1)$-dimensional wave map is established in [9], [19].

Existence of large data solutions blowing up in finite time in dimensions $n \geq 3$ is shown in [24], [4].

Stability of a trivial constant wave map and geodesic wave maps is in [27] and stability of certain $(2+1)$-dimensional spherically symmetric solutions is in [18].

For the critical $(2+1)$-dimensional wave map problem existence of solutions blowing up in finite time was proved in [23] for the $\mathbb{S}^2$ target. The large data global existence result is conjectured for the $\mathbb{H}^2$ target.

We should also mention that good results have been obtained for the large data critical $(2+1)$-dimensional wave map problem for solutions with additional *spherical* or *equivariant* symmetry assumptions. It was shown in [7] (for geodesically convex targets), [29] that large data global spherically symmetric solutions can be extended globally and uniquely in time.

The k-equivariant (co-rotational) solutions of the wave map problem are considered in the case when a target manifold is a surface of revolution. The results in [26] and [28] imply that a solution blows up in finite time only if the energy concentrates (in particular small energy implies regularity), blow-up can not occur at a self-similar rate and at the blow-up a harmonic map can be "bubbled off". We note that in the case of the $\mathbb{S}^2$ target the equation for a k-equivariant wave map takes the form

$$\partial_t^2 u - (\partial_r^2 + \frac{1}{r}\partial_r)u + k^2\,\frac{\sin(2u)}{2r^2} = 0$$

for a single scalar function u satisfying the boundary conditions $u(0) = 0$ and $u(\infty) = \pi$.