Astérisque
331, 2010, p. 317–353

EXTENSIONS FOR SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{Q}_p)$

by

Vytautas Paškūnas

Abstract. — Let $p > 2$ be a prime number. Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and π, τ smooth irreducible representations of G on $\overline{\mathbb{F}}_p$-vector spaces with a central character. We show if π is supersingular then $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ implies $\tau \cong \pi$ and compute the dimension of $\mathrm{Ext}^1_G(\pi, \pi)$. This answers affirmatively for $p > 2$ a question of Colmez. We also determine $\mathrm{Ext}^1_G(\tau, \pi)$, when π is the Steinberg representation. As a consequence of our results combined with those already in the literature one knows the extensions between all the irreducible representations of G.

***Résumé* (Extensions aux représentations supersingulières de $\mathrm{GL}_2(\mathbb{Q}_p)$).** — Soit $p > 2$ un nombre premier. Soient $G := \mathrm{GL}_2(\mathbb{Q}_p)$ et π, τ des représentations lisses irréductibles de G sur des $\overline{\mathbb{F}}_p$-espaces vectoriels avec caractère central. Nous montrons que si π est supersingulière alors $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ implique $\tau \cong \pi$ et nous calculons la dimension de $\mathrm{Ext}^1_G(\pi, \pi)$. Cela répond par l'affirmative pour $p > 2$ à une question de Colmez. Nous déterminons aussi $\mathrm{Ext}^1_G(\tau, \pi)$, quand π est la représentation de Steinberg. En conséquence de nos résultats, combinés avec ceux de la litérature, nous connaissons maintenant les extensions entre toutes les représentations irréductibles de G.

1. Introduction

In this paper we study the category Rep_G of smooth representations of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ on $\overline{\mathbb{F}}_p$-vector spaces. Smooth irreducible $\overline{\mathbb{F}}_p$-representations of G with a central character have been classified by Barthel-Livne [**1**] and Breuil [**4**]. A smooth irreducible representation π of G is supersingular, if it is not a subquotient of any principal series representation. Roughly speaking a supersingular representation is an $\overline{\mathbb{F}}_p$-analog of a supercuspidal representation.

Theorem 1.1. — *Assume that $p > 2$ and let τ and π be irreducible smooth representations of G admitting a central character. If π is supersingular and $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ then $\tau \cong \pi$. Moreover, if $p \geq 5$ then $\dim \mathrm{Ext}^1_G(\pi, \pi) = 5$.*

2010 Mathematics Subject Classification. — 22E50, 11F70.
Key words and phrases. — Supersingular, mod p Langlands correspondence.

This answers affirmatively for $p > 2$ a question of Colmez, see the introduction of [**7**]. When $p = 3$ there are two cases and we can show that in one of them $\dim \mathrm{Ext}^1_G(\pi, \pi) = 5$, in the other $\dim \mathrm{Ext}^1_G(\pi, \pi) \le 6$, which is the expected dimension. We note that if τ is a twist of Steinberg representation by a character or irreducible principal series then Colmez [**7**, VII.5.3] and Emerton [**8**, Prop. 4.3.14] prove by different methods that $\mathrm{Ext}^1_G(\tau, \pi) = 0$. Our result is new when τ is supersingular or a character.

We now explain the strategy of the proof. We first get rid of the extensions coming from the centre Z of G. Let $\zeta : Z \to \overline{\mathbb{F}}_p^\times$ be the central character of π, and let $\mathrm{Rep}_{G,\zeta}$ be the full subcategory of Rep_G consisting of representations with the central character ζ. We show in Theorem 8.1 that if $\mathrm{Ext}^1_G(\tau, \pi) \neq 0$ then τ also admits a central character ζ. Let $\mathrm{Ext}^1_{G,\zeta}(\tau, \pi)$ parameterise all the isomorphism classes of extensions between π and τ admitting a central character ζ. We show that if $\tau \not\cong \pi$ then $\mathrm{Ext}^1_{G,\zeta}(\tau, \pi) \cong \mathrm{Ext}^1_G(\tau, \pi)$ and there exists an exact sequence:

$$0 \to \mathrm{Ext}^1_{G,\zeta}(\pi, \pi) \to \mathrm{Ext}^1_G(\pi, \pi) \to \mathrm{Hom}(Z, \overline{\mathbb{F}}_p) \to 0, \tag{1}$$

where Hom denotes continuous group homomorphisms. Let I be the 'standard' Iwahori subgroup of G, (see §2), and I_1 the maximal pro-p subgroup of I. Since ζ is smooth, it is trivial on the pro-p subgroup $I_1 \cap Z$, hence we may consider ζ as a character of ZI_1. Let $\mathscr{H} := \mathrm{End}_G(\text{c-Ind}_{ZI_1}^G \zeta)$ be the Hecke algebra, and $\mathrm{Mod}_{\mathscr{H}}$ the category of right $\mathscr{H}$-modules. Let $\mathscr{I} : \mathrm{Rep}_{G,\zeta} \to \mathrm{Mod}_{\mathscr{H}}$ be the functor

$$\mathscr{I}(\kappa) := \kappa^{I_1} \cong \mathrm{Hom}_G(\text{c-Ind}_{ZI_1}^G \zeta, \kappa).$$

Vignéras shows in [**18**] that $\mathscr{I}$ induces a bijection between irreducible representations of G with the central character ζ and irreducible $\mathscr{H}$-modules. Using results of Ollivier [**13**] we show that there exists an E_2-spectral sequence:

$$\mathrm{Ext}^i_{\mathscr{H}}(\mathscr{I}(\tau), \mathbb{R}^j \mathscr{I}(\pi)) \Longrightarrow \mathrm{Ext}^{i+j}_{G,\zeta}(\tau, \pi). \tag{2}$$

The 5-term sequence associated to (2) gives an exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\tau), \mathscr{I}(\pi)) \to \mathrm{Ext}^1_{G,\zeta}(\tau, \pi) \to \mathrm{Hom}_{\mathscr{H}}(\mathscr{I}(\tau), \mathbb{R}^1 \mathscr{I}(\pi)). \tag{3}$$

Now $\mathrm{Ext}^1_{\mathscr{H}}(\mathscr{I}(\tau), \mathscr{I}(\pi))$ has been determined in [**6**] and in fact is zero if $\tau \not\cong \pi$. The problem is to understand $\mathbb{R}^1 \mathscr{I}(\pi)$ as an $\mathscr{H}$-module.

We have two approaches to this. Results of Kisin [**10**] imply that the dimension of $\mathrm{Ext}^1_G(\pi, \pi)$ is bounded below by the dimension of $\mathrm{Ext}^1_{\mathscr{G}_{\mathbb{Q}_p}}(\rho, \rho)$, where ρ is the 2-dimensional irreducible $\overline{\mathbb{F}}_p$-representation of $\mathscr{G}_{\mathbb{Q}_p}$, the absolute Galois group of $\mathbb{Q}_p$, corresponding to π under the mod p Langlands, see [**5**], [**7**]. (Excluding one case when $p = 3$.) Let $\mathfrak{I}$ be the image of $\mathrm{Ext}^1_{G,\zeta}(\pi, \pi) \to \mathrm{Hom}_{\mathscr{H}}(\mathscr{I}(\pi), \mathbb{R}^1 \mathscr{I}(\pi))$. Using (1) and (3) we obtain a lower bound on the dimension of $\mathfrak{I}$. By forgetting the $\mathscr{H}$-module structure we obtain an isomorphism of vector spaces:

$$\mathbb{R}^1 \mathscr{I}(\pi) \cong H^1(I_1/Z_1, \pi),$$

where Z_1 is the maximal pro-p subgroup of Z. The key idea is to bound the dimension of $H^1(I_1/Z_1, \pi)$ from above and use this to show if $\mathscr{I}(\tau)$ was a submodule of $\mathbb{R}^1 \mathscr{I}(\pi)$ for some $\tau \not\cong \pi$ then this would force the dimension of $\mathfrak{I}$ to be smaller than calculated before.

At the time of writing (an n-th draft of) this, [**10**] was not written up and there were some technical issues with the outline of the argument in the introductions of [**7**] and [**9**], caused by the fact that all the representations in [**7**] are assumed to have a central character. Since we only need a lower bound on the dimension of $\mathrm{Ext}^1_G(\pi, \pi)$ and only in the supersingular case, we have written up the proof of a weaker statement in the appendix. The proof given there is a variation on Colmez-Kisin argument.

In order to bound the dimension of $H^1(I_1/Z_1, \pi)$ we prove a new result about the structure of supersingular representations of G. Let M be the subspace of π generated by π^{I_1} and the semi-group $\left(\begin{smallmatrix} p^{\mathbb{N}} & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. One may show that M is a representation of I.

Theorem 1.2. — *The map $(v, w) \mapsto v - w$ induces an exact sequence of I-representations:*

$$0 \to \pi^{I_1} \to M \oplus \Pi \centerdot M \to \pi \to 0,$$

where $\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$.

We show that the restrictions of M and M/π^{I_1} to $I \cap U$, where U is the unipotent upper triangular matrices, are injective objects in $\mathrm{Rep}_{I\cap U}$. If $\psi : I \to \overline{\mathbb{F}}_p^{\times}$ is a smooth character and $p > 2$, using this, we work out $\mathrm{Ext}^1_{I/Z_1}(\psi, M)$ and $\mathrm{Ext}^1_{I/Z_1}(\psi, M/\pi^{I_1})$. Theorem 1.2 enables us to determine $H^1(I_1/Z_1, \pi)$ as a representation of I, see Theorem 7.9 and Corollary 7.10. Once one has this it is quite easy to work out $\mathbb{R}^1 \mathscr{I}(\pi)$ as an $\mathscr{H}$-module in the regular case, see Proposition 10.5, without using Colmez's work. It is also possible to work out directly the $\mathscr{H}$-module structure of $\mathbb{R}^1 \mathscr{I}(\pi)$ in the Iwahori case. However, the proof relies on heavy calculations of $\mathrm{Ext}^1_K(\mathbf{1}, \pi)$ and $\mathrm{Ext}^1_K(St, \pi)$, where $K := \mathrm{GL}_2(\mathbb{Z}_p)$ and St is the Steinberg representation of $K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_p)$. So we decided to exclude it and use "stratégie de Kisin" instead.

The primes $p = 2$, $p = 3$ require some special attention. Theorem 1.2 holds when $p = 2$, but our calculation of $H^1(I_1/Z_1, \pi)$ breaks down for the technical reason that the trivial character is the only smooth character of I, when $p = 2$. However, if $p = 2$ and we fix a central character ζ then there exists only one supersingular representation (up to isomorphism) with central character ζ. Hence, it is enough to show that $\mathrm{Ext}^1_G(\tau, \pi) = 0$ when τ is a character, since all the other cases are handled in [**7**, VII.5.3], [**8**, §4]. It might be easier to do this directly.

Let Sp be the Steinberg representation of G. After the first draft of this paper, it was pointed out to me by Emerton that it was not known (although expected) that $\mathrm{Ext}^1_G(\eta, \mathrm{Sp}) = 0$, when $\eta : G \to \overline{\mathbb{F}}_p^{\times}$ is a smooth character of order 2 (all the other cases have been worked out in [**8**, §4], see also [**7**, §VII.4,§VII.5]). A slight modification of our proof for supersingular representations also works for the Steinberg representation. In the last section we work out $\mathrm{Ext}^1_G(\tau, \mathrm{Sp})$ for all irreducible τ, when $p > 2$. As a result of this and the results already in the literature ([**6**], [**7**], [**8**]), one knows $\mathrm{Ext}^1_G(\tau, \pi)$ for all irreducible τ and π, when $p > 2$. We record this in the last section.

Acknowledgements. — The key ideas of this paper stem from a joint work with Christophe Breuil [**6**]. I would like to thank Pierre Colmez for pointing out this problem to me and Florian Herzig for a number of stimulating discussions. I would like to thank Gaëtan Chenevier, Pierre Colmez and Mark Kisin for some very helpful discussion on the "stratégie de Kisin" outlined in [**7**] and [**9**]. This paper was written when I was visiting Université Paris-Sud and IHÉS, supported by Deutsche Forschungsgemeinschaft. I would like to thank these institutions.

2. Notation

Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$, let P be the subgroup of upper-triangular matrices, T the subgroup of diagonal matrices, U be the unipotent upper triangular matrices and $K := \mathrm{GL}_2(\mathbb{Z}_p)$. Let $\mathfrak{p} := p\mathbb{Z}_p$ and

$$I := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ \mathfrak{p} & \mathbb{Z}_p^\times \end{pmatrix}, \quad I_1 := \begin{pmatrix} 1+\mathfrak{p} & \mathbb{Z}_p \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}, \quad K_1 := \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}.$$

For $\lambda \in \mathbb{F}_p$ we denote the Teichmüller lift of λ to $\mathbb{Z}_p$ by $[\lambda]$. Set

$$H := \left\{ \begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} : \lambda, \mu \in \mathbb{F}_p^\times \right\}.$$

Let $\alpha : H \to \overline{\mathbb{F}}_p^\times$ be the character

$$\alpha(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}) := \lambda\mu^{-1}.$$

Further, define

$$\Pi := \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\lambda \in \overline{\mathbb{F}}_p^\times$ we define an unramified character $\mu_\lambda : \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$, by $x \mapsto \lambda^{\mathrm{val}(x)}$.

Let Z be the centre of G, and set $Z_1 := Z \cap I_1$. Let $G^0 := \{g \in G : \det g \in \mathbb{Z}_p^\times\}$ and set $G^+ := ZG^0$.

Let $\mathscr{G}$ be a topological group. We denote by $\mathrm{Hom}(\mathscr{G}, \overline{\mathbb{F}}_p)$ the continuous group homomorphism, where the additive group $\overline{\mathbb{F}}_p$ is given the discrete topology. If $|||$ is a representation of $\mathscr{G}$ and S is a subset of $|||$ we denote by $\langle \mathscr{G} \centerdot S \rangle$ the smallest subspace of $|||$ stable under the action of $\mathscr{G}$ and containing S. Let $\mathrm{Rep}_{\mathscr{G}}$ be the category of smooth representations of $\mathscr{G}$ on $\overline{\mathbb{F}}_p$-vector spaces. If $\mathscr{Z}$ is the centre of $\mathscr{G}$ and $\zeta : \mathscr{Z} \to \overline{\mathbb{F}}_p^\times$ is a smooth character then we denote by $\mathrm{Rep}_{\mathscr{G},\zeta}$ the full subcategory of $\mathrm{Rep}_{\mathscr{G}}$ consisting of representations with central character ζ.

All the representations in this paper are on $\overline{\mathbb{F}}_p$-vector spaces.

3. Irreducible representations of K

We recall some facts about the irreducible representations of K and introduce some notation. Let σ be an irreducible smooth representation of K. Since K_1 is an open pro-p subgroup of K, the space of K_1-invariants σ^{K_1} is non-zero, and since K_1 is normal in K, σ^{K_1} is a non-zero K-subrepresentation of σ, and since σ is irreducible we obtain $\sigma^{K_1} = \sigma$. Hence the smooth irreducible representations of K coincide with the irreducible representations of $K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_p)$, and so there exists a uniquely determined pair of integers (r, a) with $0 \le r \le p-1$, $0 \le a < p-1$, such that

$$\sigma \cong \mathrm{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det{}^a .$$

Note that $r = \dim \sigma - 1$ and throughout the paper given σ, r will always mean $\dim \sigma - 1$. The space of I_1-invariants σ^{I_1} is 1-dimensional and so H acts on σ^{I_1} by a character $\chi_\sigma = \chi$. Explicitly,

$$\chi(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}) = \lambda^r (\lambda\mu)^a.$$

We define an involution $\sigma \mapsto \tilde{\sigma}$ on the set of isomorphism classes of smooth irreducible representations of K by setting

$$\tilde{\sigma} := \mathrm{Sym}^{p-r-1} \overline{\mathbb{F}}_p^2 \otimes \det{}^{r+a} .$$

Note that $\chi_{\tilde{\sigma}} = \chi_\sigma^s$. For the computational purposes it is convenient to identify $\mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ with the space of homogeneous polynomials in $\overline{\mathbb{F}}_p[x, y]$ of degree r. The action of $\mathrm{GL}_2(\mathbb{F}_p)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(x, y) := P(ax + cy, bx + dy).$$

With this identification σ^{I_1} is spanned by x^r.

Lemma 3.1. — *Let $0 \le j \le r$ be an integer and define $f_j \in \mathrm{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det^a$ by*

$$f_j := \sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1-j} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} s x^r .$$

If $r = p-1$ and $j = 0$ then $f_0 = (-1)^{a+1}(x^r + y^r)$, otherwise $f_j = (-1)^{a+1}\binom{r}{j} x^j y^{r-j}$.

Proof. — It is enough to prove the statement when $a = 0$, since twisting the action by $\det^a$ multiplies f_j by $(\det s)^a = (-1)^a$. We have

$$(4) \qquad f_j = \sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1-j} (\lambda x + y)^r = \sum_{i=0}^{r} \binom{r}{i} (\sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1+i-j}) x^i y^{r-i}.$$