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(967) *The Many Faces of the Subspace Theorem*

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THE MANY FACES OF THE SUBSPACE THEOREM [after Adamczewski, Bugeaud, Corvaja, Zannier...]

by **Yuri F. BILU***

And we discovered subspace. It gave us our galaxy and it gave us the universe. And we saw other advanced life. And we subdued it or we crushed it... With subspace, our empire would surely know no boundaries.

(From *The Great War* computer game)

1. INTRODUCTION

This is not a typical Bourbaki talk. A generic *exposé* on this seminar is, normally, a report on a recent seminal achievement, usually involving new technique. The principal character of this talk is the Subspace Theorem of Wolfgang Schmidt, known for almost forty years. All results I am going to talk about rely on this celebrated theorem (more precisely, on the generalization due to Hans Peter Schlickewei). Moreover, in all cases it is by far the most significant ingredient of the proof.

Of course, the last remark is not meant to belittle the work of the authors of the results I am going to speak about. Adapting the Subspace Theorem to a concrete problem is often a formidable task, requiring great imagination and ingenuity.

During the last decade the Subspace Theorem found several quite unexpected applications, mainly in the Diophantine Analysis and in the Transcendence Theory. Among the great variety of spectacular results, I have chosen several which are technically simpler and which allow one to appreciate how miraculously does the Subspace Theorem emerge in numerous situations, implying beautiful solutions to difficult problems hardly anybody hoped to solve so easily.

The three main topics discussed in this article are:

– the work of Adamczewski and Bugeaud on complexity of algebraic numbers;

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– the work of Corvaja and Zannier on Diophantine equations with power sums;
– the work of Corvaja and Zannier on integral points on curves and surfaces, and the subsequent development due to Levin and Autissier.

In particular, we give a complete proof of the beautiful theorem of Levin and Autissier (see Theorem 5.8): *an affine surface with* 4 *(or more) properly intersecting ample divisors at infinity cannot have a Zariski dense set of integral points.*

Originally, Schmidt proved his theorem for the needs of two important subjects: norm form equations and exponential Diophantine equations (including the polynomial-exponential equations and linear recurrence sequences). These "traditional" applications of the Subspace Theorem form a vast subject, interesting on its own; we do not discuss it here (except for a few motivating remarks in Section 4). Neither do we discuss the quantitative aspect of the Subspace Theorem. For this, the reader should consult the fundamental work of Evertse and Schlickewei (see [**33, 34, 55, 56, 57**] and the references therein).

Some of the results stated here admit far-going generalizations, but I do not always mention them: the purpose of this talk is to exhibit ideas rather than to survey the best known results.

In Section 2 we introduce the Subspace Theorem. Sections 3, 4 and 5 are totally independent and can be read in any order.

2. THE SUBSPACE THEOREM

In this section we give a statement of the Subspace Theorem. Before formulating it in full generality, we consider several particular cases, to make the general case more motivated.

2.1. The Theorem of Roth

In 1955, K. F. Roth [**51**] proved that algebraic numbers cannot be "well approximated" by rationals.

THEOREM 2.1 (Roth). — *Let α be an irrational algebraic number. Then for any $\varepsilon > 0$ the inequality*

$$\left|\alpha - \frac{y}{x}\right| < \frac{1}{|x|^{2+\varepsilon}}$$

has only finitely many solutions in non-zero $x, y \in \mathbb{Z}$.

This result is, in a sense, best possible, because, by the Dirichlet approximation theorem, the inequality $|\alpha - y/x| \leq |x|^{-2}$ has infinitely many solutions.

The theorem of Roth has a glorious history. Already Liouville showed in 1844 the inequality $|\alpha - y/x| \geq c(\alpha)|x|^{-n}$, where n is the degree of the algebraic number α, and used this to give first examples of transcendental numbers. However, Liouville's theorem was too weak for serious applications in the Diophantine Analysis. In 1909 A. Thue **[64]** made a breakthrough, proving that $|\alpha - y/x| \leq |x|^{-n/2-1-\varepsilon}$ has finitely many solutions. A series of refinements (the most notable being due to Siegel **[62]**) followed, and Roth made the final (though very important and difficult) step.

Kurt Mahler, who was a long proponent of p-adic Diophantine approximations, suggested to his student D. Ridout **[50]** to extend Roth's theorem to the non-archimedean domain. To state Ridout's result, we need to introduce some notation. For every prime number p, including the "infinite prime" $p = \infty$, we let $|\cdot|_p$ be the usual p-adic norm on $\mathbb{Q}$ (so that $|p|_p = p^{-1}$ if $p < \infty$ and $|2006|_\infty = 2006$), somehow extended to the algebraic closure $\bar{\mathbb{Q}}$. For a rational number $\xi = y/x$ with $\gcd(x, y) = 1$ we define its *height* by

$$H(\xi) = \max\{|x|, |y|\}. \tag{1}$$

One immediately verifies that

$$H(\xi) = \prod_p \max\{1, |\xi|_p\} = \left(\prod_p \min\{1, |\xi|_p\}\right)^{-1}, \tag{2}$$

where the products extend to all prime numbers, including the infinite prime.

Now let S be a finite set of primes, including $p = \infty$, and for every $p \in S$ we fix an algebraic number α_p. Ridout proved that for any $\varepsilon > 0$ the inequality

$$\prod_{p \in S} \min\left\{1, |\alpha_p - \xi|_p\right\} < \frac{1}{H(\xi)^{2+\varepsilon}}$$

has finitely many solutions in $\xi \in \mathbb{Q}$.

While the theorem of Roth becomes interesting only when the degree of α is at least 3, the theorem of Ridout is quite non-trivial even when the "targets" α_p are rational. Moreover, one can also allow "infinite" targets, with the standard convention $\infty - \xi = \xi^{-1}$. The following particular case of Ridout's theorem is especially useful: given an algebraic number α, a set S of prime numbers, and $\varepsilon > 0$, the inequality

$$|\alpha - \xi| < H(\xi)^{-1-\varepsilon}$$

has finitely many solutions in S-integers[(1)] ξ. To prove this, consider the theorem of Ridout with $\alpha_\infty = \alpha$ and with $\alpha_p = \infty$ for $p \neq \infty$, and apply (2).

[(1)] A rational number is called S-integer if its denominator is divisible only by the prime numbers from S.

One consequence of this result is that the decimal expansion of an algebraic number cannot have "too long" blocks of zeros. More precisely, let $0.a_1a_2\dots$ be the decimal expansion of an algebraic number, and for every n define $\ell(n)$ as the minimal $\ell \geq 0$ such that $a_{n+\ell} \neq 0$; then $\ell(n) = o(n)$ as $n \to \infty$. To show this, apply the above-stated particular case of the theorem of Ridout with $S = \{2, 5, \infty\}$. More generally, the decimal expansion of an algebraic number cannot have "too long" periodic blocks.

S. Lang extended the theorem of Roth-Ridout to approximation of algebraic numbers by the elements of a given number field. We invite the reader to consult Chapter 7 of his book [**41**] or Part D of the more recent volume [**40**] for the statement and the proof of Lang's theorem.

2.2. The Statement of the Subspace Theorem

Now we have enough motivation to state the Subspace Theorem. We begin with the original theorem of Schmidt [**58**] (see also [**59**] for a very detailed proof).

THEOREM 2.2 (W. M. Schmidt). — *Let $L_1, \dots, L_m$ be linearly independent linear forms in m variables with (real) algebraic coefficients. Then for any $\varepsilon > 0$ the solutions $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ of the inequality*

$$|L_1(\mathbf{x}) \cdots L_m(\mathbf{x})| \leq \|\mathbf{x}\|^{-\varepsilon}$$

are contained in finitely many proper linear subspaces of $\mathbb{Q}^m$. (Here $\|\mathbf{x}\| = \max_i\{|x_i|\}$.)

Putting $m = 2$, $L_1(x, y) = x\alpha - y$ and $L_2(x, y) = x$, we recover the theorem of Roth.

The theorem of Schmidt is not sufficient for many applications. One needs a non-archimedean generalization of it, analogous to Ridout's generalization of Roth's theorem. This result was obtained by Schlickewei [**52, 53**]. As in the previous section, let S be a finite set of prime numbers, including $p = \infty$, and pick an extension of every p-adic valuation to $\bar{\mathbb{Q}}$.

THEOREM 2.3 (H. P. Schlickewei). — *For every $p \in S$ let $L_{1,p}, \dots, L_{m,p}$ be linearly independent linear forms in m variables with algebraic coefficients. Then for any $\varepsilon > 0$ the solutions $\mathbf{x} \in \mathbb{Z}^m$ of the inequality*

$$\prod_{p \in S} \prod_{i=1}^{m} |L_{i,p}(\mathbf{x})|_p \leq \|\mathbf{x}\|^{-\varepsilon}$$

are contained in finitely many proper linear subspaces of $\mathbb{Q}^m$.