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(972) *The Renormalization Theorem of Ambrosio*

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ORDINARY DIFFERENTIAL EQUATIONS WITH ROUGH COEFFICIENTS AND THE RENORMALIZATION THEOREM OF AMBROSIO [after Ambrosio, DiPerna, Lions]

by **Camillo DE LELLIS**

INTRODUCTION

Consider the Cauchy problem for transport equations on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla_x u(t,x) \;=\; 0 \\ \\ u(0,x) \;=\; \overline{u}(x)\,. \end{cases} \tag{1}$$

Here $b : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth vector field, $\overline{u}$ a given smooth initial condition and u the unknown function. Smooth solutions of (1) are constant along curves $\phi : [a,b] \to \mathbb{R}^n$ solving the system of ordinary differential equations $\dot\phi(t) = b(t,\phi(t))$. Indeed, differentiating $g(t) = u(t,\phi(t))$ we find

$$\frac{dg}{dt} \;=\; \partial_t u(t,\phi(t)) + \dot\phi(t)\cdot\nabla_x u(t,\phi(t)) \;=\; \partial_t u(t,\phi(t)) + b(t,\phi(t))\cdot\nabla_x u(t,\phi(t)) \;=\; 0\,.$$

Thus, if $\Phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the one–parameter family of diffeomorphisms solving

$$\begin{cases} \partial_t \Phi(x,t) \;=\; b(t,\Phi(x,t)) \\ \\ \Phi(0,x) \;=\; x \end{cases} \tag{2}$$

and $\Phi^{-1}(t,\cdot)$ denotes the inverse of the diffeomorphism $\Phi(t,\cdot)$, then the unique solution u of (1) is given through the formula $u(t,x) = \overline{u}(\Phi^{-1}(t,x))$. This is the classical method of characteristics for transport equations. Our discussion justifies the name transport equation: the quantity u is simply "transported" along the trajectories of the ODE (2). It is therefore not surprising that these equations appear in the mathematical description of many phenomena in classical and statistical physics.

When b is Lipschitz, existence and uniqueness of solutions to (2) are given by the classical Cauchy–Lipschitz Theorem, but for less regular b this elegant and elementary picture breaks down. On the other hand, many physical phenomena lead naturally to consider transport equations where the coefficients b are discontinuous. The literature related to this kind of problems is huge and I will not try to give an account of it here. Let me just mention that in many of these problems one deals with coefficients which typically have jump discontinuities, take for instance the theory of shock waves.

It is therefore desirable to have a theory of solutions for ODEs and transport equations which allows for non–smooth coefficients. The Sobolev spaces $W^{1,p}$ (given by functions $u \in L^p$ with distributional derivatives in L^p) are probably the most popular spaces of irregular functions in partial differential equations. In their groundbreaking paper [**28**], motivated by their celebrated work on the Boltzmann equation, DiPerna and Lions introduced a theory of generalized solutions for transport equations and ODEs with Sobolev coefficients. Loosely speaking, this is done at the loss of a "pointwise" point of view into an "almost everywhere" point of view. Though a generic function $u \in W^{1,p}(\Omega)$ might be extremely irregular, its singular set, at least in a suitable measure theoretic sense, has necessarily codimension higher than 1. In particular, functions with jump discontinuities do not belong to $W^{1,p}$. Indeed, if the discontinuities are along nice regular surfaces, the distributional derivatives are nothing more than Radon measures.

A commonly used functional–analytic closure of such "jump functions" is the BV space, i.e. the set of summable functions whose distributional derivatives are Radon measures. The extention of the DiPerna–Lions theory to BV functions has been for a while an important open problem. After some attempts by other authors leading to partial results (see [**33**], [**15**], [**21**]; some of these works were motivated by specific problems in partial differential equations and mathematical physics), Ambrosio solved the problem in its full generality in [**4**]. This note is an attempt to illustrate the most important ideas of the DiPerna–Lions theory and of Ambrosio's result. In order to focus on the main points, I will not consider the most general results proved so far. Moreover, I will not follow the shortest proofs and often I will consider cases which later on become corollaries of more general theorems.

In the first section, I discuss the first key idea of [**28**]: the notion of renormalized solutions and its link to the uniqueness and stability for (1). In Section 2, I discuss the hard core of the DiPerna–Lions theory for $W^{1,p}$ fields: the so called commutator estimate. In Section 3, following the ideas of Ambrosio, I push gradually the DiPerna–Lions approach towards the BV case. The proof of Ambrosio's Theorem is finally achieved in Section 4 in two different ways, based on observations of Bouchut and Alberti. Section 5 discusses the third key idea of [**28**], a sort of converse of the classical theory of characteristics: appropriate results on transport equations can be used to infer interesting conclusions on ODEs. Section 6 surveys further results, conjectures and open problems in three different directions of research. Section 7 contains the proof of one technical proposition on BV functions used in Section 3.

1. RENORMALIZED SOLUTIONS

1.1. Distributional solutions

Let us start by rewriting (1) in the following way:

$$\begin{cases} \partial_t u + \operatorname{div}_x(ub) - u\operatorname{div}_x b = 0 \\ \\ u(0,x) = \overline{u}(x)\,. \end{cases} \tag{3}$$

Here and in what follows I denote by $\operatorname{div}_x b$ the divergence (in space) of the vector b. Clearly any classical solution of (3) is a solution of (1) and viceversa. However, equation (3) can be understood in the distributional sense under very mild assumptions on u and b. This is stated more precisely in the following definition.

Definition 1.1. — *Let b and $\overline{u}$ be locally summable functions such that the distributional divergence of b is locally summable. We say that $u \in L^\infty_{\rm loc}$ is a distributional solution of* (3) *if the following identity holds for every test function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$*

$$\int_0^\infty \int_{\mathbb{R}^n} u\,[\partial_t\varphi + b\cdot\nabla_x\varphi + \varphi\operatorname{div}_x b]\,dx\,dt = -\int_{\mathbb{R}^n} \overline{u}(x)\varphi(0,x)\,dx\,. \tag{4}$$

Of course for classical solutions the identity (4) follows from a simple integration by parts. The existence of weak solutions under quite general assumptions is an obvious corollary of the maximum principle for transport equations combined with a standard approximation argument.

Lemma 1.2 (Maximum Principle). — *Let b be smooth and let u be a smooth solution of* (3)*. Then, for every t we have $\sup_{x\in\mathbb{R}^n} u(t,x) \le \sup_{x\in\mathbb{R}^n}\overline{u}(x)$ and $\inf_{x\in\mathbb{R}^n} u(t,x) \ge \inf_{x\in\mathbb{R}^n}\overline{u}(x)$. Hence $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le \|\overline{u}\|_\infty$.*

Proof. — The lemma is a trivial consequence of the method of characteristics. Indeed, arguing as in the introduction $u(t,x) = \overline{u}(\Phi^{-1}(t,x))$, where Φ is the solution of (2). From this representation formula the inequalities follow trivially. □

Theorem 1.3. — *Let $b \in L^p$ with $\operatorname{div}_x b \in L^1_{loc}$ and let $\overline{u} \in L^\infty$. Then there exists a distributional solution of* (3)*.*

Proof. — Consider a standard family of mollifiers ζ_ε and η_ε respectively on $\mathbb{R}^n$ and $\mathbb{R}\times\mathbb{R}^n$. Let $b_\varepsilon = b * \eta_\varepsilon$ and $\overline{u}_\varepsilon = \overline{u} * \zeta_\varepsilon$ be the corresponding regularizations of b and $\overline{u}$. Then $\|\overline{u}_\varepsilon\|_\infty$ is uniformly bounded. Consider the classical solutions u_ε of

$$
\left\{
\begin{array}{rcl}
\partial_t u_\varepsilon + b_\varepsilon\cdot\nabla_x u_\varepsilon &=& 0\\
\\
u_\varepsilon(0,\cdot) &=& \overline{u}_\varepsilon\,.
\end{array}
\right. \tag{5}
$$

Note that such solutions exist because we can solve the equation with the method of characteristics: indeed each b_ε is Lipschitz and we can apply the classical Cauchy–Lipschitz theorem to solve (2). By Lemma 1.2 we conclude that $\|u_\varepsilon\|_\infty$ is uniformly bounded. Hence there exists a subsequence converging weakly* to a function $u\in L^\infty(\mathbb{R}^+\times\mathbb{R}^n)$. Let us fix a test function $\varphi\in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$. Since the u_ε are classical solutions of (5), the identity (4) is satisfied if we replace u, b and $\overline{u}$ with u_ε, b_ε and $\overline{u}_\varepsilon$. On the other hand, since $b_\varepsilon\to b$, $\mathrm{div}_x b_\varepsilon\to\mathrm{div}_x b$ and $\overline{u}_\varepsilon\to\overline{u}$ locally strongly in L^1_{loc}, we can pass into the limit in such identities to achieve (4) for u, $\overline{u}$ and b. □

1.2. Renormalized solutions

Of course the next relevant questions are whether such distributional solutions are unique and stable. Under the general assumptions above, the answer is negative, as it is for instance witnessed by the elegant example of **[27]**. However, DiPerna and Lions in **[28]** proved stability and uniqueness when $b\in W^{1,p}\cap L^\infty$ and $\mathrm{div}_x b\in L^\infty$.

THEOREM 1.4. — *Let $b\in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n))\cap L^\infty$ with bounded divergence. Then for every $\overline{u}\in L^\infty$ there exists a unique distributional solution of* (3). *Moreover, let b_k and $\overline{u}_k$ be two smooth approximating sequences converging strongly in L^1_{loc} to b and $\overline{u}$ such that $\|\overline{u}_k\|_\infty$ is uniformly bounded. Then the solutions u_k of the corresponding transport equations converge strongly in L^1_{loc} to u.*

In order to understand their proof, we first go back to classical solutions u of (3), and we observe that, whenever $\beta:\mathbb{R}\to\mathbb{R}$ is a C^1 function, $\beta(u)$ solves

$$
\left\{
\begin{array}{l}
\partial_t[\beta(u)] + \mathrm{div}_x[\beta(u)b] - \beta(u)\,\mathrm{div}_x b \;=\; 0\\
[\beta(u)] \;=\; \beta(\overline{u})\,.
\end{array}
\right. \tag{6}
$$

This can be seen, for instance, using the chain rule for differentiable functions, i.e. $\partial_t\beta(u) + b\cdot\nabla_x\beta(u) = \beta'(u)[\partial_t u + b\cdot\nabla_x u]$. Otherwise, one can observe that, since u must be constant along the trajectories (2), so must be $\beta(u)$. Motivated by this observation, we introduce the following terminology.

DEFINITION 1.5. — *Let $b\in L^1_{\mathrm{loc}}$ with $\mathrm{div}_x b\in L^1_{\mathrm{loc}}$. A bounded distributional solution of* (3) *is said* renormalized *if $\beta(u)$ is a solution of* (6) *for any $\beta\in C^1$. The field b is*