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(974) *Algebraization of codimension one Webs*

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ALGEBRAIZATION OF CODIMENSION ONE WEBS
[after Trépreau, Hénaut, Pirio, Robert, ...]

by **Jorge Vitório PEREIRA**

Jean-Marie Trépreau, extending previous results by Bol and Chern-Griffiths, proved recently that codimension one webs with sufficiently many abelian relations are after a change of coordinates projectively dual to algebraic curves when the ambient dimension is at least three.

In sharp contrast, Luc Pirio and Gilles Robert, confirming a guess of Alain Hénaut, independently established that a certain planar 9-web is exceptional in the sense that it admits the maximal number of abelian relations and is non-algebraizable. After that a number of exceptional planar k-webs, for every $k \geq 5$, have been found by Pirio and others.

I will briefly review the subject history, sketch Trépreau's proof, describe some of the "new" exceptional webs and discuss related recent works.

Disclaimer. — This text does not pretend to survey all the literature on web geometry but to provide a bird's-eye view over the results related to codimension one webs and their abelian relations. For instance I do not touch the interface between web geometry and loops, quasi-groups, Poisson structures, singular holomorphic foliations, complex dynamics, singularity theory, ... For more information on these subjects the reader should consult [**6, 2, 27**] and references there within.

Acknowledgements. — There are a number of works containing introductions to web geometry that I have freely used while writing this text. Here I recognize the influence of [**4, 15, 30**] and specially [**40**], which was my main source of historical references. I have also profited from discussions with C. Favre, H. Movasati, L. Pirio, F. Russo and P. Sad.

1. INTRODUCTION

A *germ of regular codimension one k-web* $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ on $(\mathbb{C}^n, 0)$ is a collection of k germs of smooth codimension one holomorphic foliations subjected to the condition that for any number m of these foliations, $m \leq n$, the corresponding tangent spaces at the origin have intersection of codimension m. Two webs $\mathcal{W}$ and $\mathcal{W}'$ are *equivalent* if there exists a germ of bihomolorphic map sending the foliations defining $\mathcal{W}$ to the ones defining $\mathcal{W}'$. Similar definitions can be made for webs of arbitrary (and even mixed) codimensions. Although most of the magic can be (and has already been) spelled in the $\mathcal{C}^{\infty}_{\mathbb{R}}$-category throughout I will restrict myself to the holomorphic category.

1.1. The origins

According to the first lines of [**6**] web geometry had its birth at the beaches of Italy in the years of 1926-27 when Blaschke and Thomsen realized that the configuration of three foliations of the plane has local invariants, see Figure 1.

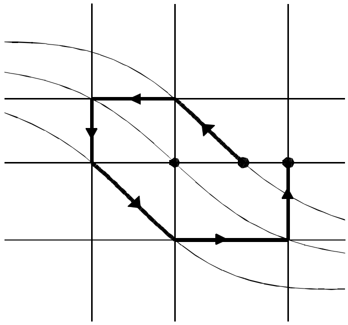

FIGURE 1. Following the leaves of foliations one obtains germs of diffeomorphisms in one variable whose equivalence class is a local invariant of the web. The web is called *hexagonal* if all the possible germs are the identity.

A more easily computable invariant was later introduced by Blaschke and Dubourdieu. If $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ is a planar web and the foliations $\mathcal{F}_i$ are defined by 1-forms ω_i satisfying $\omega_1 + \omega_2 + \omega_3 = 0$ then a simple computation shows that there exists an unique 1-form γ such that $d\omega_i = \gamma \wedge \omega_i$ for $i = 1, 2, 3$. Albeit the 1-form γ does depend on the choice of the ω_i its differential $d\gamma$ is intrinsically attached to $\mathcal{W}$, and is the so called *curvature* $\kappa(\mathcal{W})$ of $\mathcal{W}$.

Some early emblematic results of the theory developed by Blaschke and his collaborators are collected in the theorem below.

THEOREM 1.1. — *If $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ is a 3-web on $(\mathbb{C}^2, 0)$ then are equivalent:*

1. *$\mathcal{W}$ is hexagonal;*
2. *the 2-form $\kappa(\mathcal{W})$ vanishes identically;*
3. *there exists closed 1-forms η_i defining $\mathcal{F}_i$, $i = 1, 2, 3$, such that $\eta_1 + \eta_2 + \eta_3 = 0$;*
4. *$\mathcal{W}$ is equivalent to the web defined by the level sets of the functions x, y and $x - y$.*

Most of the results discussed in this text can be naïvely understood as attempts to generalize Theorem 1.1 to the broader context of arbitrary codimension one k-webs.

1.2. Abelian relations

The condition (3) in Theorem 1.1 suggests the definition of the *space of abelian relations* $\mathcal{A}(\mathcal{W})$ for an arbitrary k-web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. If the foliations $\mathcal{F}_i$ are induced by integrable 1-forms ω_i then

$$\boxed{\mathcal{A}(\mathcal{W}) = \left\{ \big(\eta_i\big)_{i=1}^{k} \in (\Omega^1(\mathbb{C}^n, 0))^k \;\middle|\; \forall i \; d\eta_i = 0\,,\ \eta_i \wedge \omega_i = 0 \ \text{ and } \sum_{i=1}^{k} \eta_i = 0 \right\}.}$$

If $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are local submersions defining the foliations $\mathcal{F}_i$ then, after integration, the abelian relations can be read as functional equations of the form $\sum_{i=1}^{k} g_i(u_i) = 0$ for some germs of holomorphic functions $g_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

Clearly $\mathcal{A}(\mathcal{W})$ is a vector space and its dimension is commonly called the *rank* of $\mathcal{W}$, denoted by $\mathrm{rk}(\mathcal{W})$. It is a theorem of Bol that the rank of a planar k-web is bounded from above by $\frac{1}{2}(k-1)(k-2)$. This bound was later generalized by Chern in his thesis (under the direction of Blaschke) for codimension one k-webs on $\mathbb{C}^n$ and reads as

$$\boxed{\mathrm{rk}(\mathcal{W}) \leq \pi(n, k) = \sum_{j=1}^{\infty} \max(0, k - j(n-1) - 1)\,.} \tag{1}$$

A k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$ is of *maximal rank* if $\mathrm{rk}(\mathcal{W}) = \pi(n, k)$. The integer $\pi(n, k)$ is the well-known Castelnuovo's bound for the arithmetic genus of irreducible and non-degenerated degree k curves in $\mathbb{P}^n$.

To establish these bounds first notice that $\mathcal{A}(\mathcal{W})$ admits a natural filtration

$$\mathcal{A}(\mathcal{W}) = \mathcal{A}^0(\mathcal{W}) \supseteq \mathcal{A}^1(\mathcal{W}) \supseteq \cdots \supseteq \mathcal{A}^j(\mathcal{W}) \supseteq \cdots,$$

where

$$\boxed{\mathcal{A}^j(\mathcal{W}) = \ker \left\{ \mathcal{A}(\mathcal{W}) \longrightarrow \left(\frac{\Omega^1(\mathbb{C}^n, 0)}{\mathfrak{m}^j \cdot \Omega^1(\mathbb{C}^n, 0)} \right)^k \right\},}$$

with $\mathfrak{m}$ being the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_n\}$.

If the submersions u_i defining $\mathcal{F}_i$ have linear term ℓ_i, then

$$\dim \frac{\mathcal{A}^j(\mathcal{W})}{\mathcal{A}^{j+1}(\mathcal{W})} \leq k - \dim\left(\mathbb{C}\cdot\ell_1^{j+1} + \cdots + \mathbb{C}\cdot\ell_k^{j+1}\right). \tag{2}$$

Since the right-hand side is controlled by the inequality, cf. [**49**, Lemme 2.1],

$$k - \dim\left(\mathbb{C}\cdot\ell_1^{j+1} + \cdots + \mathbb{C}\cdot\ell_k^{j+1}\right) \leq \max(0, k-(j+1)(n-1)-1)$$

the bound (1) follows at once. Note that this bound is attained if, and only if, the partial bounds (2) are also attained. In particular,

$$\dim \mathcal{A}(\mathcal{W}) = \pi(n,k) \implies \dim \frac{\mathcal{A}^0(\mathcal{W})}{\mathcal{A}^2(\mathcal{W})} = 2k - 3n + 1. \tag{3}$$

It will be clear at the end of the next section that the appearance of Castelnuovo's bounds in web geometry is far from being a coincidence.

1.3. Algebraizable webs and Abel's Theorem

If C is a non-degenerated[(1)] reduced degree k algebraic curve on $\mathbb{P}^n$ then for every generic hyperplane H_0 a germ of codimension one k-web $\mathcal{W}_C$ is canonically defined on $(\check{\mathbb{P}}^n, H_0)$ by projective duality. This is the web induced by the levels of the holomorphic maps $p_i : (\check{\mathbb{P}}^n, H_0) \to C$ characterized by

$$H \cdot C = p_1(H) + p_2(H) + \cdots + p_k(H)$$

for every H sufficiently close to H_0.

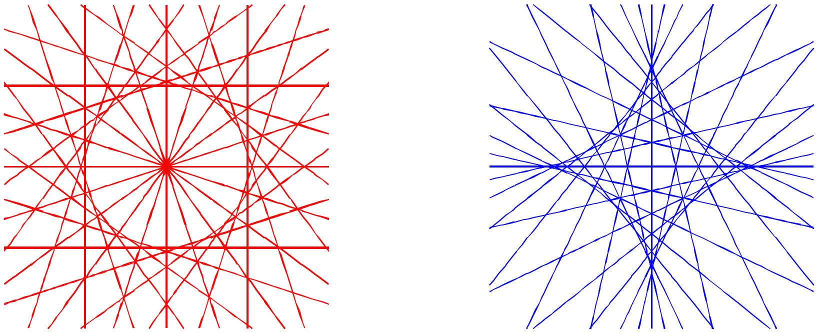

FIGURE 2. On the left $\mathcal{W}_C$ is pictured for a reduced cubic curve C formed by a line and a conic. On the right $\mathcal{W}_C$ is drawn for a rational quartic C.

(1) Throughout the term non-degenerated will be used in a stronger sense than usual in order to ensure that the dual web is smooth. It means that any collection of points in the intersection of C with a generic hyperplane, but not spanning the hyperplane, is formed by linearly independent points.