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SEMI-CLASSICAL MEASURES AND ENTROPY [after Nalini Anantharaman and Stéphane Nonnenmacher]

by **Yves COLIN de VERDIÈRE**

INTRODUCTION

This report is about recent progress on semi-classical localization of eigenfunctions for quantum systems whose classical limit is hyperbolic (Anosov systems); the main example is the Laplace operator on a compact Riemannian manifold with strictly negative curvature whose classical limit is the geodesic flow; the quantizations of hyperbolic cat maps, called "quantum cat maps", are other nice examples. All this is part of the field called "quantum chaos". The new results are:

- Examples of eigenfunctions for the cat maps with a strong localization ("scarring") effect due to S. de Bièvre, F. Faure and S. Nonnenmacher [**17, 16**].
- Uniform distribution of Hecke eigenfunctions in the case of arithmetic Riemann surfaces by E. Lindenstrauss [**26**].
- General lower bounds on the entropy of semi-classical measures due to N. Anantharaman [**1**] and improved by N. Anantharaman–S. Nonnenmacher [**3**] and N. Anantharaman–H. Koch–S. Nonnenmacher [**2**]. This lower bound is sharp with respect to the cat maps examples.

We will mainly focus on this last result.

1. THE 2 BASIC EXAMPLES

1.1. Cat maps

We start with a matrix $A \in SL_2(\mathbb{Z})$ which is assumed to be hyperbolic: the eigenvalues $\lambda_\pm$ of A satisfy $0 < |\lambda_-| < 1 < |\lambda_+|$. The action of A onto $\mathbb{R}^2$ defines a symplectic action U of A on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by considering action on points mod $\mathbb{Z}^2$.

Such a map is a simple example of a chaotic map. It has been observed since a long time that such a map can be quantized: for each integer N, we consider the Hilbert space $\mathscr{H}_N$ of dimension N of Schwartz distributions f which are periodic of period one and of which Fourier coefficients are periodic of period N: if $f(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx}$, we have, for all $k \in \mathbb{Z}$, $a_{k+N} = a_k$. Using the metaplectic representation applied to A, we get a natural unitary action $\hat{U}_N$ onto the space $\mathscr{H}_N$. We are mainly interested in the eigenfunctions of $\hat{U}_N$. The semi-classical parameter is $\hbar = 1/N$ and the classical limit corresponds to large values of N. A good reference is [**8**].

1.2. The Laplace operators

On a smooth compact connected Riemannian manifold (X, g) without boundary, we consider the Laplace operator Δ given in local coordinates by

$$\Delta = -|g|^{-1}\partial_i g^{ij}|g|\partial_j$$

with $|g| = \det(g_{ij})$. The Laplace operator Δ is essentially self-adjoint on $L^2(X)$ with domain the smooth functions and has a compact resolvent. The spectrum is discrete and denoted by

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

with an orthonormal basis of eigenfunctions φ_k satisfying $\Delta\varphi_k = \lambda_k\varphi_k$. It is useful to introduce an effective Planck constant (the semi-classical small parameter) $\hbar := \lambda_k^{-\frac{1}{2}}$. We will rewrite the eigenfunction equation $\hbar^2\Delta\varphi = \varphi$. The semi-classical limit $\hbar \to 0$ corresponds to the high frequency limit for the periodic solutions $u(x,t) = \exp(i\sqrt{\lambda_k}t)\varphi_k$ of the wave equation $u_{tt} + \Delta u = 0$. Instead of the wave evolution, we will use the Schrödinger evolution which is given by

$$\frac{\hbar}{i}u_t = -\frac{\hbar^2}{2}\Delta u \ ,$$

and introduce the unitary dynamics defined by the 1-parameter group

$$\hat{U}^t = \exp(-it\hbar\Delta/2), \ t \in \mathbb{R}.$$

For the basic definitions, one can read [**5**].

1.3. The geodesic flow

If (X, g) is a Riemannian manifold and $v \in T_xX$ a tangent vector at the point $x \in X$, we define, for $t \in \mathbb{R}$, $G^t(x,v) = (y,w)$ as follows: if $\gamma(t)$ is the geodesic which satisfies $\gamma(0) = x$, $\dot\gamma(0) = v$, we put $y := \gamma(t)$ and $w := \dot\gamma(t)$. By using the identification of the tangent bundle with the cotangent bundle induced by the metric g (which is also the Legendre transform of the Lagrangian $\frac{1}{2}g_{ij}(x)v_iv_j$), we get a flow $(G^t)^\star$ on $T^\star X$ which preserves the unit cotangent bundle denoted by Z. We denote by U^t the restriction of $(G^t)^\star$ to Z. The *Liouville measure* dL on Z is the Riemannian measure normalized as a probability measure. The Liouville measure dL is invariant by the geodesic flow.

2. CLASSICAL CHAOS

Good textbooks on the classical chaos are [**21, 28, 10**].

2.1. Classical Hamiltonian systems

We consider a closed phase space Z which is the torus $\mathbb{R}^2/\mathbb{Z}^2$ in the case of the cat map and the unit cotangent bundle in the case of the Laplace operator. On Z, we have the Liouville measure dL which is normalized as a probability measure. Moreover, we have a measure preserving smooth dynamics on Z which is the action of U in the cat map example and the geodesic flow in the Riemannian case. We will denote this action by U^t where t belongs to $\mathbb{Z}$ or to $\mathbb{R}$.

2.2. Ergodicity

DEFINITION 2.1. — *The dynamical system* (Z, U^t, dL) *is* ergodic *if every measurable set which is invariant by* U^t *is of measure* 0 *or* 1*.*

As a consequence, we get the celebrated *Birkhoff ergodic Theorem:*

THEOREM 2.2. — *If* (Z, U^t, dL) *is ergodic, for every* $f \in L^1(Z, dL)$ *and almost every* $z \in Z$*:*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(U^t z) dt = \int_Z f dL \ .$$

The cat map is ergodic and the geodesic flow of every closed Riemannian manifold with < 0 sectional curvature is ergodic too.

2.3. Mixing

A much stronger property is the *mixing property* which says that we have a correlation decay:

DEFINITION 2.3. — *The dynamical system* U^t *is* mixing *if for every* f, $g \in L^2(Z, dL)$ *with* $\int_Z f dL = 0$*, we have*

$$\lim_{t \to \infty} \int_Z f(U^t(z)) g(z) dL = 0 \ .$$

Cat maps as well as geodesic flows on manifolds with < 0 curvature are mixing. Mixing systems are ergodic.

2.4. Liapounov exponent

Chaotic systems are often presented as (deterministic) dynamical systems which are very sensitive to initial conditions.

DEFINITION 2.4. — *The global Liapounov exponent* Λ_+ *of the smooth dynamical system* (Z, U^t) *is defined as the lower bounds of the* Λ*'s for which the differential* dU^t *of the dynamics satisfies*

$$\|dU^t(z)\| = O(e^{\Lambda t}) \ ,$$

for $t \to +\infty$*, uniformly w.r. to* z.

For cat maps given by A, $\Lambda_+ = \log |\lambda_+|$. If X is a Riemannian manifold of sectional curvature -1, $\Lambda_+ = 1$.

2.5. K-S entropy

Kolmogorov and Sinaï start from the work of Shannon in information theory in order to introduce an entropy $h_{\mathrm{KS}}(\mu)$ for a dynamical system with an invariant probability measure μ. The definition of the entropy uses partitions of the phase space and how they are refined by the dynamics:

DEFINITION 2.5. — *If* $\mathscr{P} = \{\Omega_j | j = 1, \cdots, N\}$ *is a finite measurable partition of* Z*, we define the* entropy $h(\mathscr{P}) := -\sum \mu(\Omega_j) \log \mu(\Omega_j)$.

In terms of information theory, it is the average information you get by knowing in which of the Ω_j's the point z lies. Let $\mathscr{P}^{\vee N}$ be the partition whose sets are

$$\Omega_{j_1, j_2, \cdots, j_N} = \{z \in Z \text{ so that, for } l = 1, \cdots, N+1, \ U^{l-1}(z) \in \Omega_{j_l}\} \ .$$

If we define $\mathscr{P}_1 \vee \mathscr{P}_2$ as the partition whose elements are the intersections of one element of the partition $\mathscr{P}_1$ and one element of the partition $\mathscr{P}_2$, we get from the properties of the log function:

$$h(\mathscr{P}_1 \vee \mathscr{P}_2) \leq h(\mathscr{P}_1) + h(\mathscr{P}_2) \ .$$

Let us define $\mathscr{P}_1 = \mathscr{P}^{\vee n}$ and $\mathscr{P}_2 = U^{-n}(\mathscr{P}^{\vee m})$. Using the *invariance*[1] *of* μ *by* U, we get $h(\mathscr{P}_2) = h(\mathscr{P}^{\vee m})$. From $\mathscr{P}^{\vee (n+m)} = \mathscr{P}_1 \vee \mathscr{P}_2$, we get the *sub-additivity* of the sequence $N \to h(\mathscr{P}^{\vee N})$.

We define

$$h_{\mathrm{KS}}(\mathscr{P}) := \lim_{N \to \infty} h(\mathscr{P}^{\vee N})/N \ ,$$

and $h_{\mathrm{KS}}(\mu) = \sup_{\mathscr{P}} h_{\mathrm{KS}}(\mathscr{P})$.

(1) The invariance of μ is used in a crucial way here and, as we will see, it is one of the problem we have to solve when passing to the quantum case.