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UNFOLDINGS OF TANGENT TO THE IDENTITY DIFFEOMORPHISMS

by

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Abstract. — This paper is devoted to classify one dimensional unfoldings of tangent to the identity analytic diffeomorphisms, in other words elements φ of $\mathrm{Diff}(\mathbb{C}^2, 0)$ of the form $(x, f(x, y))$ with $(\partial f/\partial y)(0, 0) = 1$. We provide the topological classification in absence of small divisors phenomena and an analytic classification of the finite codimension unfoldings. Such results are based on the study of the stable structures preserved by the diffeomorphisms. The main tool is the use of real flows. In both the topological and the analytic cases a non-wandering property is required, namely the Rolle property in the topological setting and infinitesimal stability in the analytic one.

We also prove that under generic hypotheses the analytic class of an unfolding φ depends only on the analytic classes of the germs of 1-dimensional diffeomorphisms obtained by localizing along an irreducible component of the fixed points set of φ.

***Résumé* (Déploiements des difféomorphismes tangents à l'identité).** — Cet article est consacré à la classification des déploiements à un paramètre des difféomorphismes analytiques tangents à l'identité, en d'autres termes, les éléments φ de $\mathrm{Diff}(\mathbb{C}^2, 0)$ qui sont de la forme $(x, f(x, y))$, avec $(\partial f/\partial y)(0, 0) = 1$. Nous fournissons une classification topologique en l'absence de phénomènes de petits diviseurs et la classification analytique des déploiements de codimension finie. Les preuves sont basées sur l'étude des structures stables qui sont invariants par l'action des difféomorphismes. L'outil principal est le recours aux flots réels. Une propriété de non-errance est nécessaire, à savoir la propriété de Rolle dans le cas topologique et la stabilité infinitésimale dans le cas analytique.

On prouve aussi que, sous des hypothèses génériques, la classe analytique d'un déploiement φ ne dépend que des classes analytiques des germes de difféomorphismes en dimension 1 obtenus en localisant le long d'une composante irréductible de l'ensemble des points fixes de φ.

1. Introduction

We classify one dimensional unfoldings of tangent to the identity diffeomorphisms, i.e. elements φ of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$ of the form $(x, f(x, y))$ with $(\partial f/\partial y)(0, 0) = 1$. The

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set of diffeomorphisms of the previous form is denoted by $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. We provide a topological classification of the multi-parabolic elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ and an analytic classification of the non-degenerate ones (see section 2). Such results are based on the study of the stable structures preserved by the diffeomorphisms whose main tool is the use of real flows (section 3). This part of the paper is a survey of the papers **[10]** (topological classification) and **[12]** (analytic classification).

In section 6 we present a new result. We are interested on the local behavior of global objects. For instance in our setting we are interested on describing the nature of $\varphi = (x, f(x,y)) \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ in a neighborhood of $\gamma \setminus \{(0,0)\}$ for an irreducible component γ of the fixed points set $\mathrm{Fix}(\varphi)$ of φ. Given $(x_0,y_0) \in \gamma \setminus \{(0,0)\}$ we denote by $\varphi_{(x_0,y_0)}$ the germ of $\varphi_{|x=x_0}$ in the neighborhood of $y = y_0$. Suppose γ is parabolic, i.e. $(\partial f/\partial y)_{|\mathrm{Fix}(\varphi)} \equiv 1$. Then "part" of the Ecalle-Voronin invariants of $\varphi_{(x,y)}$, where (x,y) belongs to $\gamma \setminus \{(0,0)\}$, can be extended continuously to $x = 0$ in good sectors S in the parameter space. Under the proper hypothesis (see theorem 6.1) we can prove that the analytic class of φ in the neighborhood of $\gamma \setminus \{(0,0)\}$ determines the analytic class of φ in $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. By varying the parameter x we show that the Ecalle Voronin invariants of $\varphi_{(x,y)}$ "turn" with respect to the Ecalle Voronin invariants of φ (see subsection 6.2). Section 6 is a small glimpse of a more detailed work to be published.

2. Notations

We denote by $\mathcal{X}(\mathbb{C}^2,0)$ the group of germs of complex analytic vector fields in a neighborhood of $0 \in \mathbb{C}^2$. The elements of $\mathcal{X}(\mathbb{C}^2,0)$ which are singular at $0 \in \mathbb{C}^2$ can be interpreted as derivations of the maximal ideal of the ring $\mathbb{C}\{x,y\}$. We denote by $\hat{\mathcal{X}}(\mathbb{C}^2,0)$ the group of derivations of the maximal ideal $\hat{\mathfrak{m}}$ of the ring $\mathbb{C}[[x,y]]$. An element $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}^2,0)$ can be expressed in the more conventional form

$$\hat{X} = \hat{X}(x)\frac{\partial}{\partial x} + \hat{X}(y)\frac{\partial}{\partial y}.$$

Let $\mathrm{Diff}\,(\mathbb{C}^2,0)$ be the group of germs of complex analytic diffeomorphisms in a neighborhood of $0 \in \mathbb{C}^2$. We define $\mathrm{Fix}(\varphi)$ the *fixed points set* of $\varphi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$. We denote by $\mathrm{Diff}_1(\mathbb{C}^2,0)$ the subgroup of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ whose elements are tangent to the identity, i.e. given $\varphi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$ then it belongs to $\mathrm{Diff}_1(\mathbb{C}^2,0)$ if $j^1\varphi \equiv Id$. Let $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ be the formal completion of $\mathrm{Diff}\,(\mathbb{C}^2,0)$.

We denote by $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ the subgroup of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ of unfoldings of tangent to the identity diffeomorphisms. More precisely an element $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form

$$\varphi(x,y) = (x, f(x,y))$$

where $(\partial f/\partial y)(0,0) = 1$. We say that $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is *non-degenerate* if $y \circ \varphi(0,y) \not\equiv y$. The linear part $j^1\varphi$ of an element φ of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form $(x, y + ax)$ for some $a \in \mathbb{C}$. The linear unipotent isomorphism $(x, y + ax)$ is the exponential of the linear nilpotent vector field $ax\partial/\partial y$. Indeed φ is the exponential of a unique nilpotent formal vector field $\hat{X}$. More precisely φ can be interpreted as an operator $g \to g \circ \varphi$ acting on $\hat{\mathfrak{m}}$. The operator φ is the exponential of an operator $\hat{X}$ acting on $\hat{\mathfrak{m}}$ as a derivation and such that $j^1 \hat{X} = ax\partial/\partial y$. We say that $\hat{X}$ is the *infinitesimal generator* of φ. We denote $\log \varphi = \hat{X}$.

Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$; we say that φ is *multi-parabolic* if $(\partial f/\partial y)_{|\mathrm{Fix}(\varphi)} \equiv 1$. We denote by $\mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ the set of multi-parabolic unfoldings and we call its elements MP-diffeomorphisms.

Given $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ we denote $\varphi \sim_{\mathrm{an}} \eta$ if φ and η are conjugated by some $\sigma \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $x \circ \sigma = x$ and $\sigma_{|\mathrm{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$. If we replace $\mathrm{Diff}(\mathbb{C}^2, 0)$ with the group of germs of homeomorphisms we obtain the equivalence $\varphi \sim_{\mathrm{top}} \eta$. By replacing $\mathrm{Diff}(\mathbb{C}^2, 0)$ with $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ we obtain $\varphi \sim_{for} \eta$. In the formal setting $\sigma_{|\mathrm{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$ means that $y \circ \sigma - y$ belongs to the ideal of $\overline{\mathrm{Fix}(\varphi) \setminus \{x=0\}}$ in the ring $\mathbb{C}[[x, y]]$ (supposed $x \circ \sigma = x$).

3. Real flows

Our goal is describing the dynamics of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Instead of trying a direct approach we consider a continuous dynamical system similar to φ. More precisely we choose a germ of holomorphic vector field $X_\varphi = g(x, y)\partial/\partial y$ satisfying the *proximity condition*, namely

$$y \circ \varphi - y \circ \exp(X_\varphi) \in (y \circ \varphi - y)^2.$$

Such a choice of X_φ is possible **[11]** even if not unique. Supposed $\varphi = \exp(X_\varphi)$ then the orbits of φ would be contained in the trajectories of the real flow $\Re(X_\varphi)$ of X_φ. Anyway $\Re(X_\varphi)$ provides a continuous "model" for the iterates of φ.

The proximity condition implies that $\mathrm{Fix}(\varphi) = Sing X_\varphi$ and that φ is formally conjugated to $\exp(X_\varphi)$ **[11]**.

Definition 3.1. — *We say that φ satisfies the ϵ-property if there exist open neighborhoods $V \subset W$ of $(0,0)$ such that for all $(x, y) \in V$ and $j \in \mathbb{Z}$ satisfying*

$$\cup_{k \in [\min(j,0), \max(j,0)] \cap \mathbb{Z}} \{\exp(X_\varphi)^k (x, y)\} \subset V$$

then $\cup_{k \in [\min(j,0), \max(j,0)] \cap \mathbb{Z}} \{\varphi^k (x, y)\} \subset W$ and

$$\varphi^j (x, y) \in \exp(B(0, \epsilon) X_\varphi)(\exp(X_\varphi)^j (x, y)).$$

We say that φ satisfies the stability property if it satisfies the ϵ-property for any $\epsilon > 0$ small enough.

***Theorem 3.1* (ϵ-theorem or stability theorem)**. — **[10]** *Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Then φ satisfies the stability property for every choice of X_φ satisfying the proximity property.*

The ϵ-theorem implies that in the multi-parabolic case the orbits of φ and $\exp(X_\varphi)$ remain close independently of the number of iterations. Moreover, the dynamics of φ is roughly speaking the dynamics of $\exp(X_\varphi)$ plus some small "noise". Analyzing the noise is not trivial since not every MP-diffeomorphism is topologically conjugated to the exponential of a holomorphic vector field as we will see later on. The stability property is crucial [**10**] to provide a complete system of topological invariants for the MP-diffeomorphisms. The situation in the general case is different since

Theorem 3.2. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0) \setminus \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$. Then φ holds the ϵ-property for some choice of X_φ if and only if $\log\varphi \in \mathcal{X}(\mathbb{C}^2,0)$.*

Proof. — Denote $\varphi(x,y) = (x, f(x,y))$. Since $\varphi \notin \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ there exists an irreducible component γ_0 of $\mathrm{Fix}(\varphi)$ such that $(\partial(y\circ\varphi)/\partial y)_{|\gamma_0} \not\equiv 1$. Up to replace φ with $(x, f(x^k,y))$ and γ_0 with one of the irreducible components of $(x^k,y)^{-1}(\gamma_0)$ for some k in $\mathbb{N}$ we can choose γ_0 of the form $y = h(x)$. Up to the change of coordinates $(x, y-h(x))$ we can suppose $\gamma_0 \equiv \{y=0\}$. Denote $L(w) = (\partial(y\circ\varphi)/\partial y)(w,0)$.

Let $w \in \mathbb{C}$; denote φ_w the germ of $\varphi_{|x=w}$ in the neighborhood of $(w,0)$. Suppose $L(w) \in \mathbb{S}^1 \setminus \{1\}$, then the ϵ-property implies that the sequence $\{\varphi_w^j\}$ is normal in some neighborhood of $(w,0)$. Therefore φ_w is analytically linearizable for any $w \in L^{-1}(\mathbb{S}^1\setminus\{1\})$ and then for any w in a pointed neighborhood of 0 in $\mathbb{C}$.

The infinitesimal generator $\log\varphi$ is of the form $\hat{f}(x,y)\partial/\partial y$ for some $\hat{f} \in \mathbb{C}[[x,y]]$. We have $\hat{f} = \sum_{j\geq 0} f_j(x)y^j$. Indeed $\hat{f}$ is transversally formal along γ_0, i.e. there exists a neighborhood $V \subset \mathbb{C}$ of 0 such that $f_j \in \mathcal{O}(V)$ for any $j \geq 0$. This is a consequence of φ_w being linearizable for any $w \in L^{-1}(e^{2\pi i\mathbb{Q}} \setminus \{1\})$ [**11**].

Consider a path $\eta \subset V \setminus \{0\}$ turning once around 0 and transversal to $L^{-1}(\mathbb{S}^1)$. Moreover we can suppose that whenever $w \in \eta \cap L^{-1}(\mathbb{S}^1)$ then $L(w)$ is a Bruno number. Denote by $\sigma(w,y)$ the element of $\mathrm{Diff}_1(\mathbb{C},0)$ linearizing φ_w. By the choice of η then σ is continuous in $\eta\times W_0$ for some neighborhood W_0 of $0\in\mathbb{C}$. As a consequence $\hat{f}$ is a continuous function in $\eta\times W$ for some neighborhood W of $0 \in \mathbb{C}$. Then there exists $C \in \mathbb{R}^+$ such that

$$|f_j(w)| \leq C^j \ \text{ for all } (w,j) \in \eta\times\mathbb{N}.$$

The modulus maximum principle implies that $\log\varphi \in \mathcal{X}(\mathbb{C}^2,0)$. □

The previous theorem implies that the dynamics of a generic $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is unstable. Then the stable dynamics of a vector field is not a good model of the dynamics of φ. As a consequence the study of real flows of holomorphic vector fields is no good to classify topologically generic elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.

In spite of the previous discussion real flows are useful to provide a complete system of analytic invariants for non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Why this? This phenomenon is linked to the rigidity of analytic structures. Roughly speaking by doing cuts in the domain of definition of φ we can find subsets S such that the dynamics of $\varphi_{|S}$ is stable and close to the dynamics of $\exp(X_\varphi)_{|S}$. Moreover, the analytic class of φ is determined by the analytic classes of $\varphi_{|S}$ for good choices of S (here the rigidity in the analytic world plays a role). The cuts in the domain of definition allows us to avoid the instability related to resonances, small divisors and renormalized return maps. This point of view is developed in section 5 to obtain the theorem of analytic classification.

Theorem 3.3. — *Consider non-degenerate elements* φ, η *of* $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. *Suppose that* $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$. *Then* $\varphi \sim_{\mathrm{an}} \eta$ *if and only if there exists* $r \in \mathbb{R}^+$ *such that for any* w *in a pointed neighborhood of* 0 *the restrictions* $\varphi_{|x=w}$ *and* $\eta_{|x=w}$ *are conjugated by an injective holomorphic mapping defined in* $B(0,r)$ *and fixing the points in* $\mathrm{Fix}(\varphi) \cap \{x = w\}$.

In the previous theorem the mappings conjugating the restrictions of φ and η to $x = w$ do not depend a priori continuously on w. Even so we can obtain an analytic conjugation because the spaces of orbits associated to the cuts $\varphi_{|S}$ are rigid (see section 5). The theorem is representative of a more general property: a non-degenerate element φ of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ inherits the rigidity properties associated to $\varphi_{|x=0}$ and its space of orbits.

Resuming, we can use real flows of holomorphic vector fields to catch the stable structures contained in the dynamics of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. We will use such information to classify topologically stable elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ and to classify analytically non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$.

$$\text{topological stability} \;\rightarrow\; \text{topological classification}$$

$$\text{analytic substability} \;+ \text{rigidity of analytic structures} \;\rightarrow\; \text{analytic classification.}$$

4. Topological classification of MP-diffeomorphisms

Consider $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Because of the stability property its dynamics by iteration is close to the dynamics of $\exp(X_\varphi)$. The latter one is embedded in the dynamics of the vector field $\Re(X_\varphi)$. The subsections 4.1, 4.2 and 4.4 are intended to describe briefly the dynamics of $\Re(X_\varphi)_{|x=w}$ for w in a neighborhood of $0 \in \mathbb{C}$.

In subsection 4.5 we introduce the tools to describe the instability phenomena attached to $\Re(X_\varphi)$, we also explain that the dynamics of $\Re(X_\varphi)_{|x=w}$ is simple for generic values of w. That could make us think that the dynamics of $\Re(X_\varphi)_{|x=w}$ does not depend on w. In subsection 4.6 we show that this is not the case if $\sharp(\mathrm{Fix}(\varphi) \cap \{x = w\}) > 1$ for $w \neq 0$.

The subsection 4.7 is intended to describe the properties of the sets Z corresponding to unstable parameters. The limit of the dynamics of $\Re(X_\varphi)_{|x=w}$ when $w \in Z$ and $w \to 0$ is more complicated than the dynamics of $\Re(X_\varphi)_{|x=0}$. In order to do this, given an analytic curve $\gamma \subset Z$ and a sequence of points $\gamma \times \mathbb{C} \ni (w_n, y_n) \to (0, y_0) \neq (0,0)$, we study the limits of the trajectories of $\Re(X_\varphi)$ passing through points (w_n, y_n) when $n \to \infty$. There are choices of γ and (w_n, y_n) such that the limit is bigger than the closure of the trajectory of $\Re(X_\varphi)$ passing through $(0, y_0)$. In subsection 4.8 we describe the evolution of the limits of trajectories with respect to γ. The existence of big limits (or long trajectories) is invariant by topological conjugation, their study provides the first topological invariants both for vector fields (subsection 4.9) and diffeomorphisms (subsection 4.10). These invariants are of formal type, they only depend on the formal class of the vector field or diffeomorphism. A second type