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GEVREY CLASS OF THE INFINITESIMAL GENERATOR OF A DIFFEOMORPHISM

by

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Abstract. — Let F be an analytic diffeomorphism in $(\mathbb{C}^m, 0)$ tangent to the identity of order n. The infinitesimal generator of F is the formal vector field X such that $\operatorname{Exp} X = F$. In this paper we provide an elementary proof of the fact that X belongs to the Gevrey class of order $1/n$.

Résumé **(La classe de Gevrey du générateur infinitésimal d'un difféomorphisme)**
Soit F un difféomorphisme analytique de $\mathbb{C}^m$ tangent à l'identité à l'ordre n. Le générateur infinitésimal de F est le champ de vecteurs formel X tel que $\operatorname{Exp} X = F$. Dans cet article nous donnons une preuve élémentaire du fait que X appartient à la classe Gevrey d'ordre $1/n$.

1. Introduction

For each couple of integers $m \geq 1$ and $n \geq 2$, let us denote $\hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ the module of formal vector fields of order $\geq n$ in $(\mathbb{C}^m, 0)$ and $\widehat{\operatorname{Diff}}_n(\mathbb{C}^m, 0)$ the group of formal diffeomorphisms in $(\mathbb{C}^m, 0)$ tangent to the identity of order $\geq n$, i.e, $F \in \widehat{\operatorname{Diff}}_n(\mathbb{C}^m, 0)$ if and only if $\nu(F) := \min\{\nu_0(x_i \circ F - x_i) | i = 1, \ldots, m\} - 1 \geq n$. For any $X \in \widehat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$, the exponential operator of X is the application $\exp X : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by the formula

$$\exp X(g) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(g)$$

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where $X^0(g) = g$ and $X^{j+1}(g) = X(X^j(g))$. It is a classical result (for instance, see [**5**]) that the application

$$\begin{array}{rcl} \mathrm{Exp} : \widehat{\mathfrak{X}}_n(\mathbb{C}^m, 0) & \to & \widehat{\mathrm{Diff}}_{n-1}(\mathbb{C}^m, 0) \\ X & \mapsto & (\exp X(x_1), \ldots, \exp X(x_m)) \end{array}$$

is a bijection. The formal vector field X such that $F = \mathrm{Exp}(X)$ is called the *infinitesimal generator* of F.
Let $x = (x_1, \ldots, x_m)$ and for any $s \in \mathbb{R}$ let $\mathbb{C}[[x]]_s$ denote the subset of elements of $\mathbb{C}[[x]]$ that satisfy the s-Gevrey condition, i.e.

$$f(x) = \sum_{k=0}^{\infty} f_k(x) \in \mathbb{C}[[x]]_s \quad \text{if and only if} \quad \sum_{k=0}^{\infty} \frac{f_k(x)}{k!^s} \in \mathbb{C}\{x\},$$

where $f_k(x)$ is homogeneous of degree k. Let us observe that 0-Gevrey condition means analyticity, and $\mathbb{C}\{x\} \subset \mathbb{C}[[x]]_s \subset \mathbb{C}[[x]]_t$ if $0 < s < t$. Let $\mathfrak{X}_n(\mathbb{C}^m, 0)_s \subseteq \hat{\mathfrak{X}}_n(\mathbb{C}^m, 0)$ be the set of s-Gevrey vector fields $X = \sum_{k=1}^{m} X(x_k) \frac{\partial}{\partial x_k}$ with $X(x_k) \in \mathbb{C}[[x]]_s$ and $\mathrm{Diff}_n(\mathbb{C}^m, 0)_s = \widehat{\mathrm{Diff}}_n(\mathbb{C}^m, 0) \cap (\mathbb{C}[[x]]_s)^m$ the set of s-Gevrey diffeomorphisms tangent to the identity of order $\geq n$.
We will prove the following result

Theorem 1.1. — *For any $s \geq \frac{1}{n-1}$ the application* Exp *gives a bijection*

$$\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_s \to \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s.$$

In particular, the infinitesimal generator of any tangent to the identity analytic diffeomorphism F is $\frac{1}{\nu(F)}$-Gevrey.

In general, X may be divergent for a convergent F, for instance, Szekeres [**7**] and Baker [**2**] proved that every entire holomorphic function tangent to the identity of order k in dimension 1 has a non-convergent infinitesimal generator, Ahern and Rosay [**1**] proved that this kind of diffeomorphisms cannot be the time-1 map of a C^{3k+3}-vector field, and finally J. Rey [**6**] showed that they cannot be the time-1 map of a C^{k+1}-vector field, which is the best possible bound. Thus, the map $\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_0 \to \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_0$ is not surjective for any couple of positive integers m, n. In addition, in dimension 1, using resummation arguments, it is proved that if an analytic diffeomorphism $f(x) = x + a_{k+1}x^{k+1} + \cdots$ with $a_{k+1} \neq 0$ has a divergent infinitesimal generator X, then X is k-summable, so X is Gevrey of order $\frac{1}{k}$, but not smaller (see [**4**], [**3**] and [**5**]). Therefore, the condition $s \geq \frac{1}{n-1}$ is necessary.

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2. Technical estimations

In this paper, we take the following notations:

- $h_k(x)$ will denote the homogeneous polynomial $\sum\limits_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=k}} x^\alpha$.
- $H_{s,n}(x)$ the series $\sum\limits_{q=n}^{\infty}(q+m-n)!^s h_q(x)$.
- $\frac{\partial}{\partial x}$ the differential operator $\sum\limits_{k=1}^{m}\frac{\partial}{\partial x_k}$.

For formal series $f(x)=\sum_\alpha f_\alpha x^\alpha$ and $g(x)=\sum_\alpha g_\alpha x^\alpha$, we say that $f\preceq g$ if $|f_\alpha|\le|g_\alpha|$ for any $\alpha\in\mathbb{N}^m$. We get in this way a partial order in $\mathbb{C}[[x]]$, and also in $\hat{\mathfrak{X}}_n(\mathbb{C}^m,0)$ and $\widehat{\mathrm{Diff}}_n(\mathbb{C}^m,0)$, working on the component function. From the definition of Gevrey condition, it can be seen that $X\in\mathfrak{X}_n(\mathbb{C}^m,0)_s$ if and only if there exists $a\in\mathbb{R}^+$ such that, for all $q\ge n$,

$$\mathrm{Coef}_q(X)\preceq(q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x},$$

where $\mathrm{Coef}_q(X)$ denotes the homogeneous term of X of degree q. Thus $X\in\mathfrak{X}_n(\mathbb{C}^m,0)_s$ if and only if there exists $a\in\mathbb{R}^+$ such that $X\preceq H_{s,n}(ax)\frac{\partial}{\partial x}$.
We need the following technical lemmas:

Lemma 2.1. — *For every $k,l\in\mathbb{N}^*$*

$$h_k\frac{\partial}{\partial x}h_l\preceq(l+m-1)\min\left\{\binom{k+m-1}{m-1},\binom{l+m-2}{m-1}\right\}h_{k+l-1}.$$

Proof. — Observe that

$$\begin{aligned}\frac{\partial}{\partial x}h_l &= \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=l}}x^\alpha=\sum_{k=1}^{m}\sum_{\substack{\alpha\in\mathbb{N}^m\\|\alpha|=l}}\alpha_k\frac{x^\alpha}{x_k}\\ &= \sum_{\substack{\beta\in\mathbb{N}^m\\|\beta|=l-1}}\sum_{k=1}^{m}(\beta_k+1)x^\beta=(l+m-1)h_{l-1}\end{aligned}$$

Now, the coefficient of x^α in the product $h_k(x)h_{l-1}(x)$ is less than or equal to the minimum between the number of monomials of h_k and the number of monomials of h_{l-1}, and the number of monomials of h_j is $\binom{j+m-1}{m-1}$, that corresponds to the number of ordered partitions of j in m parts; therefore,

$$h_k\frac{\partial}{\partial x}h_l=(l+m-1)h_kh_{l-1}\preceq(l+m-1)\binom{\min\{k,l-1\}+m-1}{m-1}h_{k+l-1}.\quad\square$$

Lemma 2.2. — *Let $\Theta(y)=\sum\limits_{j=n}^{\infty}\binom{m-1+j}{m-1}y^{j-n}$. Then $\Theta(y)$ converges for any $|y|<1$.*

Proof. — Since $\sum\limits_{j=n}^{\infty} y^{m-1+j} = \frac{y^{m+n-1}}{1-y}$ converges for any $|y|<1$ then

$$\Theta(y) = \frac{1}{(m-1)!}\frac{1}{y^n}\frac{d^{m-1}}{dy^{m-1}}\left(\frac{y^{m+n-1}}{1-y}\right)$$

converges for any $|y|<1$. □

Lemma 2.3. — *For any $s>0$ and integers $m\geq 1$ and $n\geq 2$, the sequence $\{b_q\}_{q\geq 2n-1}$ given by*

$$b_q = \sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor}\left(\frac{(j+m-n)!(q-j+1+m-n)!}{m!(q+m-n)!}(q-j+m)^{n-1}\right)^s\binom{j+m-1}{m-1},$$

is bounded.

Proof. — Observe that

$$\begin{aligned}\frac{(q-j+m)^{n-1}}{(q-j+2+m-n)\cdots(q-j+m)} &< \left(\frac{q-j+m}{q-j+2+m-n}\right)^{n-1}\\ &\leq \left(\frac{\frac{q-1}{2}+m}{\frac{q-1}{2}+2+m-n}\right)^{n-1} \leq \left(\frac{m+n-1}{m+1}\right)^{n-1}\end{aligned}$$

then

$$b_n \leq \left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor}\left(\frac{(j+m-n)!(q-j+m)!}{m!(q+m-n)!}\right)^s\binom{j+m-1}{m-1}.$$

In addition

$$\frac{m+1}{q+m-j+1} < \frac{m+2}{q+m-j+2} < \cdots < \frac{j+m-n}{q+m-n}$$

and

$$\frac{j+m-n}{q+m-n} \leq \frac{\frac{q+1}{2}+m-n}{q+m-n} \leq \max\left\{\frac{1}{2},\frac{m}{m+n-1}\right\} = C_{m,n} < 1;$$

from lemma 2.2,

$$b_q < \left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\Theta(C^s_{m,n}).$$ □

Proposition 2.4. — *Let $s\geq\frac{1}{n-1}$, $X\in\widehat{\mathfrak{X}}_n(\mathbb{C}^m,0)$ and $a\in\mathbb{R}^+$ such that*

$$\mathrm{Coef}_q(X) \preceq (q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x}$$

for all $n\leq q\leq N$, and let us denote $A = 2m!^s\left(\frac{m+n-1}{m+1}\right)^{s(n-1)}\Theta(C^s_{m,n})$. For every q,k with $n\leq q\leq N+k-1$,

$$\mathrm{Coef}_q(X^k) \preceq (aA)^{k-1}(q+m-n)!^s a^q h_q(x)\frac{\partial}{\partial x},$$

Proof. — Since $X^k = \sum_{i=1}^{m} X^k(x_i)\frac{\partial}{\partial x_i}$, it is enough to prove the affirmation for $X^k(x_i)$, where $i \in \{1, 2, \dots, m\}$. Let us write $X = \sum_{j=n}^{\infty} X_j$, where X_j is homogeneous of degree j. We will proceed by induction on k; if $k = 1$, by hypothesis

$$X_q(x_i) \preceq (q+m-n)!^s a^q h_q(x) \quad \text{for every } n \leq q \leq N.$$

Suppose that the lemma is true for every $k \leq p$, then, since the order of X^j is greater than or equal to $(n-1)j+1$, $\mathrm{Coef}_q(X^{p+1}) = 0$ for $n \leq q \leq (n-1)p+n-1$ and for $(n-1)p+n \leq q \leq N+p$ we have

$$\begin{aligned}
&\mathrm{Coef}_q(X^{p+1}(x_i)) = \mathrm{Coef}_q(X(X^p(x_i))) = \mathrm{Coef}_q\Big(\sum_{j=n}^{\infty} X_j(X^p(x_i))\Big)\\
&= \sum_{j=n}^{q-(n-1)p} X_j \,\mathrm{Coef}_{q+1-j}(X^p(x_i))\\
&\preceq \sum_{j=n}^{q-(n-1)p} (j+m-n)!^s a^j h_j(x)\frac{\partial}{\partial x}\Big((aA)^{p-1}(q-j+1+m-n)!^s a^{q+1-j} h_{q+1-j}(x)\Big)\\
&\preceq \sum_{j=n}^{q-n+1} (j+m-n)!^s (q-j+1+m-n)!^s (q-j+m)\tbinom{\min\{j,q-j\}+m-1}{m-1} A^{p-1}a^{q+p}h_q,\\
&\preceq 2\sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor} ((j+m-n)!(q-j+1+m-n)!(q+m-j)^{n-1})^s \tbinom{j+m-1}{m-1} A^{p-1}a^{q+p}h_q.
\end{aligned}$$

Now, observe that

$$b_q m!^s (q+m-n)!^s = \sum_{j=n}^{\lfloor\frac{q+1}{2}\rfloor} ((j+m-n)!(q-j+1+m-n)!(q-j+m)^{n-1})^s \binom{j+m-1}{m-1},$$

where $\{b_q\}$ is the sequence defined in lemma 2.3; it follows that

$$\begin{aligned}
\mathrm{Coef}_q(X^{p+1}(x_i)) &\preceq 2b_q m!^s (q+m-n)!^s A^{p-1}a^{q+p}h_q\\
&\preceq (q+m-n)!^s (aA)^p a^q h_q
\end{aligned}$$

□

3. Proof of theorem 1.1.

To prove that the application $\mathrm{Exp} : \mathfrak{X}_n(\mathbb{C}^m, 0)_s \to \mathrm{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ is well defined for $s \geq \frac{1}{n-1}$, let $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$, $a > 0$ be such that $X \preceq H_{s,n}(ax)$, and A as in proposition 2.4.
Then by proposition 2.4 we have

$$\begin{aligned}
\mathrm{Coef}_q(\exp X(x_j)) &= \sum_{k=1}^{\infty} \frac{1}{k!}\mathrm{Coef}_q(X^k(x_j))\\
&\preceq \sum_{k=1}^{\infty} \frac{1}{k!}(aA)^{k-1}(q+m-n)!^s a^q h_q(x)
\end{aligned}$$