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ORDINARY PARTS OF ADMISSIBLE REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS I. DEFINITION AND FIRST PROPERTIES

by

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Abstract. — If G is a connected reductive p-adic group, P is a parabolic subgroup of G, and M is a Levi factor of P, and if A is an Artinian local ring having a finite residue field of characteristic p, then we define a functor Ord_P from the category of admissible smooth P-representations over A to the category of admissible smooth M-representations of A, which we call the functor of ordinary parts. We show that this functor is right adjoint to the functor of parabolic induction $\mathrm{Ind}_{\overline{P}}^{G}$, where $\overline{P}$ is an opposite parabolic to P.

***Résumé* (Parties ordinaires de représentations admissibles de groupes réductifs p-adiques I. Définitions et premières propriétés)**

Soit G un groupe p-adique connexe réductif, P un sous-groupe parabolique de G, et M un facteur de Levi de P. Si A est un anneau local artinien ayant un corps résiduel fini de caractéristique p, alors nous définissons un foncteur Ord_P de la catégorie des P-représentations sur A lisses et admissibles vers la catégorie des M-représentations de A lisses et admissibles, que nous appelons foncteur des parties ordinaires. Nous montrons que ce foncteur est adjoint à droite du foncteur d'induction parabolique $\mathrm{Ind}_{\overline{P}}^{G}$, où $\overline{P}$ est un opposé parabolique de P.

1. Introduction

This paper is the first of two in which we define and study the functors of ordinary parts on categories of admissible smooth representations of p-adic reductive groups over fields of characteristic p, as well as their derived functors. These functors of ordinary parts, which are characterized as being right adjoint to parabolic induction, play an important role in the study of smooth representation theory in characteristic p. They also have important global applications: when applied in the context of the

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p-adically completed cohomology spaces introduced in [**6**], they provide a representation theoretic approach to Hida's theory of (nearly) ordinary parts of cohomology [**11, 12, 13**]. The immediate applications that we have in mind are for the group GL_2: the results of our two papers have applications to the construction of the mod p and p-adic local Langlands correspondences for the group $\mathrm{GL}_2(\mathbb{Q}_p)$, and to the investigation of local-global compatibility for p-adic Langlands over the group GL_2 over $\mathbb{Q}$.

To describe our results more specifically, let G be (the $\mathbb{Q}_p$-valued points of) a connected reductive p-adic group, P a parabolic subgroup of G, $\overline{P}$ an opposite parabolic to P, and $M = P \bigcap \overline{P}$ the corresponding Levi factor of P and $\overline{P}$. If k is a finite field of characteristic p, we let $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ (resp. $\mathrm{Mod}_M^{\mathrm{adm}}(k)$) denote the category of admissible smooth G-representations (resp. M-representations) over k. Parabolic induction yields a functor $\mathrm{Ind}_{\overline{P}}^G : \mathrm{Mod}_M^{\mathrm{adm}}(k) \to \mathrm{Mod}_G^{\mathrm{adm}}(k)$. The functor of ordinary parts associated to P is then a functor $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{adm}}(k) \to \mathrm{Mod}_M^{\mathrm{adm}}(k)$, which is right adjoint to $\mathrm{Ind}_{\overline{P}}^G$.

In this paper, we define Ord_P and study its basic properties. In the sequel [**10**] we investigate the derived functors of Ord_P, and present some applications to the computation of Ext spaces in the category $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ in the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, computations which in turn play a role in the construction of the mod p and p-adic local Langlands correspondences (see [**4**] and [**8**]). The applications to local-global compatibility are part of the arguments of [**8**]. We hope to discuss the applications in the context of p-adically completed cohomology in a future paper. Let us only mention here that Theorem 3.4.8 below is an abstract formulation of Hida's general principle that the ordinary part of cohomology should be finite over weight space.

1.1. Arrangement of the paper. — In order to allow for maximum flexibility in applications, we actually work throughout the paper with representations defined not just over the field k, but over general Artinian local rings, or even complete local rings, having residue field k. This necessitates a development of the foundations of the theory of admissible representations over such coefficient rings. Such a development is the subject of Section 2. We also take the opportunity in that section to present some related representation theoretic notions that do not seem to be in the literature, and which will be useful in this paper, its sequel, and future applications. The functors Ord_P are defined, and some of their basic properties established, in Section 3. Their characterization as adjoint functors is proved in Section 4. In the appendix we establish some simple functional analytic results about modules over p-adic integer rings.

1.2. Notation and terminology. — Throughout the paper, we fix a prime p, as well as a finite extension E of $\mathbb{Q}_p$, with ring of integers $\mathscr{O}$. We let $\mathbb{F}$ denote the residue field of $\mathscr{O}$, and ϖ a choice of uniformizer of $\mathscr{O}$. Let $\mathrm{Comp}(\mathscr{O})$ denote the category of complete Noetherian local $\mathscr{O}$-algebras having finite residue fields, and let $\mathrm{Art}(\mathscr{O})$

denote the full subcategory of $\mathrm{Comp}(\mathscr{O})$ consisting of those objects that are Artinian (or equivalently, of finite length as $\mathscr{O}$-modules).

If A is an object of $\mathrm{Comp}(\mathscr{O})$, and V is an A-module which is torsion as an $\mathscr{O}$-module, then we write $V^* := \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})$ to denote the Pontrjagin dual of V (where V is endowed with its discrete topology), equipped with its natural profinite topology. If V is a G-representation over A, for some group G, then the contragredient action makes V^* a G-representation over A (with each element of G acting via a continuous automorphism).

If V is any $\mathscr{O}$-module, then we let V_{fl} denote the maximal $\mathscr{O}$-torsion free quotient of V. We write $V[\varpi^i]$ to denote the kernel of multiplication by ϖ^i on V, and $V[\varpi^\infty] := \bigcup_{i\geq 0} V[\varpi^i]$. If $\mathscr{C}$ is any $\mathscr{O}$-linear category, then for any integer $i \geq 0$ (resp. $i = \infty$), we let $\mathscr{C}[\varpi^i]$ denote the full subcategory of $\mathscr{C}$ consisting of objects that are annihilated by ϖ^i (resp. by some power of ϖ).

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2. Representations of p-adic analytic groups

Let G be a p-adic analytic group. Throughout this section, we will let A denote an object of $\mathrm{Comp}(\mathscr{O})$, with maximal ideal $\mathfrak{m}$. We denote by $\mathrm{Mod}_G(A)$ the abelian category of representations of G over A (with morphisms being A-linear G-equivariant maps); equivalently, $\mathrm{Mod}_G(A)$ is the abelian category of $A[G]$-modules, where $A[G]$ denotes the group ring of G over A. In this section we introduce, and study some basic properties of, various categories of G-representations, as well as of certain related categories of what we will call augmented G-representations.

2.1. Augmented G-representations. — If H is a compact open subgroup of G, then as usual we let $A[[H]]$ denote the completed group ring of H over A, i.e.

$$A[[H]] := \varprojlim_{H'} A[H/H'], \tag{2.1.1}$$

where H' runs over all normal open subgroups of H. We equip $A[[H]]$ with the projective limit topology obtained by endowing each of the rings $A[H/H']$ appearing on the right hand side of (2.1.1) with its $\mathfrak{m}$-adic topology. The rings $A[H/H']$ are then profinite (since each H/H' is a finite group), and hence this makes $A[[H]]$ a profinite, and so in particular compact, topological ring.

2.1.2. *Theorem*. — *The completed group ring $A[[H]]$ is Noetherian.*

Proof. — In the case when $A = \mathscr{O} = \mathbb{Z}_p$, this is proved by Lazard [**14**]: after replacing H by an open subgroup, if necessary, Lazard equips $\mathbb{Z}_p[[H]]$ with an exhaustive decreasing filtration $F^\bullet$ whose associated graded ring $\mathrm{Gr}^\bullet_F\mathbb{Z}_p[[H]]$ is isomorphic to a polynomial algebra (with generators in degree 1) over the graded ring $\mathbb{F}_p[t]$ (which is the graded ring associated to $\mathbb{Z}_p$ with its p-adic filtration).

For general A, we may write $A[[H]] \xrightarrow{\sim} A\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[H]]$, and so equip $A[[H]]$ with an exhaustive decreasing filtration obtained as the completed tensor product of the $\mathfrak{m}$-adic filtration on A and Lazard's filtration $F^\bullet$ on $\mathbb{Z}_p[[H]]$. The graded ring associated to this filtration on the completed tensor product is naturally isomorphic to a quotient of the tensor product of the individual associated graded rings, i.e. is isomorphic to a quotient of $(\mathrm{Gr}^\bullet_{\mathfrak{m}}A) \otimes_{\mathbb{F}_p[t]} \mathrm{Gr}^\bullet_F\mathbb{Z}_p[[H]]$, where $\mathrm{Gr}^\bullet_{\mathfrak{m}}A$ denotes the graded ring associated to the $\mathfrak{m}$-adic filtration on A (which is also naturally an algebra over the graded ring $\mathbb{F}_p[t]$ associated to the p-adic filtration of $\mathbb{Z}_p$, since $p \in \mathfrak{m}$).

Thus the associated graded ring to $A[[H]]$ is isomorphic to a quotient of a polynomial algebra over the Noetherian ring $\mathrm{Gr}^\bullet_{\mathfrak{m}}A$, and thus is again Noetherian. Consequently $A[[H]]$ itself is Noetherian, as claimed. □

2.1.3. *Proposition*. — 1. *Any finitely generated $A[[H]]$-module admits a unique profinite topology with respect to which the $A[[H]]$-action on it becomes jointly continuous.*

2. *Any $A[[H]]$-linear morphism of finitely generated $A[[H]]$-modules is continuous with respect to the profinite topologies on its source and target given by part* (1).

Proof. — Since $A[[H]]$ is Noetherian, we may find a presentation

$$A[[H]]^s \to A[[H]]^r \to M \to 0$$

of M, for some $r, s \geq 0$. Since $A[[H]]$ is profinite, it follows that the first arrow has closed image, and hence that M is the quotient of the profinite module $A[[H]]^r$ by a closed $A[[H]]$-submodule. If we equip M with the induced quotient topology, then it becomes a profinite module. The resulting profinite topology on M is clearly independent of the chosen presentation, and satisfies the requirements of the proposition. □

2.1.4. *Definition*. — If M is a finitely generated $A[[H]]$-module, we refer to the topology given by the preceding lemma as the canonical topology on M.

2.1.5. *Definition*. — By an augmented representation of G over A we mean an $A[G]$-module M equipped with an $A[[H]]$-module structure for some (equivalently, any) compact open subgroup H of G, such that the two induced $A[H]$-actions (the first induced by the inclusion $A[H] \subset A[[H]]$ and the second by the inclusion $A[H] \subset A[G]$) coincide.

2.1.6. *Definition*. — By a profinite augmented G-representation over A we mean an augmented G-representation M over A that is also equipped with a profinite topology, so that the $A[[H]]$-action on M is jointly continuous [(1)] for some (equivalently any) compact open subgroup H over A.

[(1)] It is equivalent to ask that the profinite topology on M admits a neighborhood basis at the origin consisting of $A[[H]]$-submodules.

The equivalence of the conditions "some" and "any" in the preceding definitions follows from the fact that if H_1 and H_2 are two compact open subgroups of G, then $H := H_1 \bigcap H_2$ has finite index in each of H_1 and H_2.

We denote by $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ the abelian category of augmented G-representations over A, with morphisms being maps that are simultaneously G-equivariant and $A[[H]]$-linear for some (equivalently, any) compact open subgroup H of G, and by $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ the abelian category of profinite augmented G-representations, with morphisms being continuous A-linear G-equivariant maps (note that since $A[H]$ is dense in $A[[H]]$, any such map is automatically $A[[H]]$-linear for any compact open subgroup H of G). Forgetting the topology induces a forgetful functor

$$\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \longrightarrow \mathrm{Mod}_G^{\mathrm{aug}}(A).$$

We let $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ consisting of augmented G-modules that are finitely generated over $A[[H]]$ for some (equivalently any) compact open subgroup H of G. Proposition 2.1.3 shows that by equipping each object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ with its canonical topology, we may lift the inclusion $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \to \mathrm{Mod}_G^{\mathrm{aug}}(A)$ to an inclusion $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \to \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$.

2.1.7. *Proposition*. — *The category* $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ *forms a Serre subcategory of each of the abelian categories* $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ *and* $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ *(i.e. it is closed under passing to subobjects, quotients, and extensions). In particular, it itself is an abelian category.*

Proof. — Closure under the formation of quotients and extensions is evident, and closure under the formation of subobjects follows from Theorem 2.1.2. □

2.2. Smooth G-representations. — In this subsection we give the basic definitions, and state the basic results, related to smooth, and admissible smooth, representations of G over A. In the case when A is Artinian, our definitions will agree with the standard ones. Otherwise, they may be slightly unorthodox, but will be the most useful ones for our later purposes.

2.2.1. *Definition*. — Let V be a representation of G over A. We say that a vector $v \in V$ is smooth if:

1. v is fixed by some open subgroup of G.
2. v is annihilated by some power $\mathfrak{m}^i$ of the maximal ideal of A.

We let V_{sm} denote the subset of V consisting of smooth vectors.

2.2.2. *Remark*. — The following equivalent way of defining smoothness can be useful: a vector $v \in V$ is smooth if and only if v is annihilated by the intersection $J \bigcap A[H]$ for some open ideal $J \subset A[[H]]$, where H is some (equivalently, any) compact open subgroup of G.

2.2.3. *Remark*. — If A is Artinian, then $\mathfrak{m}^i = 0$ for sufficiently large i, and so automatically $\mathfrak{m}^i v = 0$ for any element v of any A-module V. Thus condition (2) can be