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Lou van den DRIES

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Secrétariat : Nathalie Christiaën

Astérisque

Société Mathématique de France

Institut Henri Poincaré, 11, rue Pierre et Marie Curie

75231 Paris Cedex 05, France

Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96

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APPROXIMATE GROUPS
[according to Hrushovski and Breuillard, Green, Tao]

by **Lou van den DRIES**

1. INTRODUCTION

Throughout G is an ambient group. Let $X, Y \subseteq G$, and set

$$XY := \{xy : x \in X, y \in Y\}, \qquad X^{-1} := \{x^{-1} : x \in X\},$$
$$X^0 := \{1\} \subseteq G, \quad X^1 := X, \quad X^2 := XX, \quad X^3 := XXX, \text{ and so on.}$$

Let $\langle X \rangle$ denote the subgroup of G generated by X. A *left coset* of X is a translate $gX \subseteq G$ (even if X is not a subgroup of G). We use the term *right coset* in the same way. Call X *symmetric* if $1 \in X$ and $X^{-1} = X$. Throughout, $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and K, L are real numbers ≥ 1. Note that if X is symmetric, then $\langle X \rangle = \bigcup_n X^n$. When we say that Y is covered by K left (respectively, right) cosets of X we mean that there exists $E \subseteq G$ of cardinality $|E| \leq K$ such that $Y \subseteq EX$ (respectively, $Y \subseteq XE$).

Call X an *approximate group* (in G) if X is symmetric and X^2 can be covered by finitely many left cosets of X (equivalently, by finitely many right cosets of X). Of course, this notion is trivial for finite X. Any compact symmetric neighborhood of the identity in a locally compact group is clearly an approximate group. Call X a *K-approximate group* if X is symmetric and X^2 can be covered by K left cosets of X (equivalently, by K right cosets). This notion is of particular interest when X is finite. It is easy to check that 1-approximate groups in G are subgroups of G.

We think of K as small and fixed, and are interested in the structure of finite K-approximate groups X when its cardinality $|X|$ is large compared to K. On this we have the following result due to Breuillard, Green, Tao **[2]** and much of it conjectured by H. Helfgott and also by E. Lindenstrauss:

THEOREM 1.1. — *If $X \subseteq G$ is a finite K-approximate group, then there is a K^6-approximate*[(1)] *group $Y \subseteq X^4$, such that:*

(i) *X is covered by L left cosets of Y, where L depends only on K;*
(ii) *$\langle Y\rangle$ has a d-nilpotent subgroup of finite index, with $d \leq 3\log_2 K$.*

Here a group H is called *d-nilpotent* ($d \in \mathbb{N}$) if H is generated by elements $u_1, \dots, u_d$ such that $[u_i, u_j] \in \langle u_1, \dots, u_{i-1}\rangle$ whenever $1 \leq i < j \leq d$, in particular, $u_1 \in \mathrm{Z}(H)$; note that then H is nilpotent of class $\leq d$. We also call $u_1, \dots, u_d$ a *nilpotent base* of H if the above holds.

The proof of Theorem 1.1 uses Hrushovski's modeling [**13**] of limits of finite K-approximate groups by compact neighborhoods of the identity in Lie groups. This may remind you of Gromov [**8**] on groups of polynomial growth, and among the applications of Theorem 1.1 are indeed strengthenings of Gromov's result. These are derived in Section 3, which also includes a generalized "Margulis Lemma" conjectured by Gromov; for more on this, see the paper by Courtois [**3**] in this volume.

Theorem 1.1 says that finite K-approximate groups are largely controlled by nilpotent groups. A more detailed version of this theorem in [**2**] gives even tighter control by so-called *coset nilprogressions*, which generalize symmetric arithmetic progressions in $\mathbb{Z}$. This amounts to a qualitative generalization of earlier "inverse" theorems by Freiman and Ruzsa in *additive combinatorics*, the study of set addition in abelian groups; see Tao and Van Vu [**23**]. *Multiplicative combinatorics* is its extension to arbitrary groups, and we start with some basic facts from this subject in Section 2 after sketching the proof of Theorem 1.1. That theorem, however, is trivial for finite G (take $Y = X$), unlike the detailed main result in [2]. But its proof avoids the more complicated *local* group setting of [**2**]. How to bound L in (i) explicitly in terms of K is not known. Such explicit bounds are known for various natural classes of finite groups; see Helfgott [**9, 10, 11**].

Sketch of proof for Theorem 1.1

For fixed K, finite K-approximate groups $X_i \subseteq G_i$ as $|X_i| \to \infty$ behave roughly like their (logical) limits $X \subseteq G$ where X is now a *pseudofinite* K-approximate group and the model-theoretic structure (G, X) is *rich* in a certain logical sense. (See Section 4 for the logical notions involved.) The properties of the (pseudo)counting measure on G, normalized so that X has measure 1, lead by a fundamental result in [**13**] to a group morphism $\pi : \langle X\rangle \to \mathscr{G}$ onto a locally compact group $\mathscr{G}$ with good properties such as $\ker(\pi) \subseteq X^4$.

(1) The K^6-bound is not in [**2**]. The bounds in (i) and (ii) are more important.

Yamabe's theorem on approximating locally compact groups by Lie groups permits changing π to a group morphism $\rho : \langle Y \rangle \to \mathscr{H}$ onto a connected Lie group $\mathscr{H}$ for some definable symmetric $Y \subseteq X^4$ such that $\ker(\rho) \subseteq Y$ and finitely many left cosets of Y cover X. (See Section 4 on "definability".) Let H be the smallest definable subgroup of G containing Y. We use induction on $d := \dim \mathscr{H}$ to construct definable normal subgroups H_i of H such that

$$\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{2d+1} = H$$

and the quotient H_{i+1}/H_i is pseudofinite for even i, and pseudocyclic and central in H/H_i for odd i. (On general logical grounds and by a group theoretic lemma this gives a weak version of Theorem 1.1, with bounds depending only on K instead of the specific bounds K^6 and $3\log_2 K$. The latter require additional steps.) To prepare for this induction we first use the "no small subgroups" property of Lie groups to shrink Y, without changing $\langle Y \rangle$ or H, so that the image of Y^2 in $\mathscr{H}$ contains no nontrivial subgroup of $\mathscr{H}$. Next, with $\mathbb{N}^* \supseteq \mathbb{N}$ and $\mathbb{R}^*$ in the role of $\mathbb{N}$ and $\mathbb{R}$, we define for $g \in G$ its exit norm (or escape norm) $|g| = |g|_Y \in \mathbb{R}^*$ by

$$|g| := \begin{cases} 0 & \text{if } g^\nu \in Y \text{ for all } \nu \in \mathbb{N}^*, \\ 1/\nu & \text{if } \nu \in \mathbb{N}^* \text{ is minimal with } g^\nu \notin Y. \end{cases}$$

Thus $0 \leq |g| \leq 1$, and $|g| < 1 \Leftrightarrow g \in Y$. The Lie group $\mathscr{H}$ is controlled near the identity by its Lie algebra via the exponential map, and this allows **[2]** to adapt arguments stemming from Gleason **[6]** to show that for some $C \in \mathbb{N}$ and all $g, h \in Y$ we have

$$|gh| \leq C \cdot (|g| + |h|), \qquad |ghg^{-1}| \leq C|h|, \qquad |[g,h]| \leq C \cdot |g| \cdot |h|.$$

This yields a definable normal subgroup of H, namely

$$H_1 := \{h \in H : |h| = 0\} = \{h \in H : h^\nu \in Y \text{ for all } \nu \in \mathbb{N}^*\}$$

with $H_1 \subseteq \ker(\rho) \subseteq Y$. If $d = 0$, then $H = H_1 = Y = \langle Y \rangle$ and we are done, so assume $d > 0$. Replacing H by H/H_1 and Y by its image in H/H_1 without changing $\mathscr{H}$, we arrange that $|h| > 0$ for all $h \neq 1$ in H. Since Y is pseudofinite, we have $u \in Y$ with minimal $|u| > 0$. Then $|u|$ is infinitesimal, and the bound on the exit norm of commutators $[g,h]$ yields that u lies in the center of H. Let $H_2 := u^{\mathbb{Z}^*}$ be the smallest definable subgroup of H containing u. Replacing H by H/H_2 and Y by its image in H/H_2, we can replace $\mathscr{H}$ by the lower dimensional Lie group $\mathscr{H}/\mathscr{H}_2$, where $\mathscr{H}_2$ is the closure of the central subgroup $\rho(H_2 \cap \langle Y \rangle)$ in $\mathscr{H}$. This decrease in dimension gives by induction the desired result.