369 ASTÉRISQUE

2015

DE LA GÉOMÉTRIE ALGÉBRIQUE
AUX FORMES AUTOMORPHES (I)

J.-B. BOST, P. BOYER, A. GENESTIER,
L. LAFFORGUE, S. LYSENKO, S. MOREL, B.C. NGÔ, eds.

A CATEGORICAL APPROACH TO THE STABLE CENTER CONJECTURE

Roman BEZRUKAVNIKOV, David KAZHDAN & Yakov VARSHAVSKY

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque est un périodique de la Société Mathématique de France.

Numéro 369, 2015

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 82 € ($ 123)
Abonnement Europe : 650 €, hors Europe : 689 € ($ 1033)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0303-1179

ISBN 978-2-85629-805-3

Directeur de la publication : Marc Peigné

Astérisque
369, 2015, p. 27–97

A CATEGORICAL APPROACH TO THE STABLE CENTER CONJECTURE

by

Roman Bezrukavnikov, David Kazhdan & Yakov Varshavsky

To Gérard Laumon on his 60th birthday

Abstract. — Let G be a connected reductive group over a local non-archimedean field F. The stable center conjecture provides an intrinsic decomposition of the set of equivalence classes of smooth irreducible representations of $G(F)$, which is only slightly coarser than the conjectural decomposition into L-packets. In this work we propose a way to verify this conjecture for depth zero representations. As an illustration of our method, we show that the Bernstein projector to the depth zero spectrum is stable.

***Résumé* (Une approche catégorique de la conjecture du centre stable).** — Soit G un groupe réductif connexe sur un corps local non archimédien F. La conjecture du centre stable fournit une décomposition intrinsèque de l'ensemble des classes d'équivalence de représentations lisses irréductibles de $G(F)$, qui est seulement un peu plus grossière que la décomposition en L-paquets. Nous proposons dans ce travail une voie de vérification de cette conjecture pour les représentations de profondeur zéro. À titre d'illustration de notre méthode, nous montrons que le projecteur de Bernstein vers le spectre de profondeur zéro est stable.

Introduction

The stable center conjecture. — Let G be a connected reductive group over a local non-archimedean field F, let $R(G)$ be the category of smooth complex representations of $G(F)$, and let Z_G be the Bernstein center of $G(F)$, which is by definition the center of the category $R(G)$. Then Z_G is a commutative algebra over $\mathbb{C}$.

Every $z \in Z_G$ defines an invariant distribution ν_z on $G(F)$, and we denote by Z_G^{st} the set of all $z \in Z_G$ such that the distribution ν_z is stable. The stable center conjecture asserts that Z_G^{st} is a unital subalgebra of Z_G.

2010 ***Mathematics Subject Classification.*** — Primary: 22E50; Secondary: 22E35, 22E57, 14D24.
Key words and phrases. — Local Langlands conjecture, Bernstein center, affine Weyl group, categorical Hecke algebra, infinity categories, ℓ-adic sheaves.

This conjecture is closely related to the local Langlands conjecture. Recall that the local Langlands conjecture asserts that the set of equivalence classes of smooth irreducible representations $\mathrm{Irr}(G)$ of $G(F)$ decomposes as a disjoint union of so-called L-packets. By definition, we have a natural homomorphism $z \mapsto f_z$ from Z_G to the algebra of functions $\mathrm{Fun}(\mathrm{Irr}(G), \mathbb{C})$. A more precise version of the stable center conjecture asserts that Z_G^{st} consists of all $z \in Z_G$ such that the function f_z is constant on each L-packet.

In other words, the local Langlands conjecture allows a more precise formulation of the stable center conjecture. Conversely, if $Z_G^{st} \subset Z_G$ is known to be a subalgebra, then we can decompose $\mathrm{Irr}(G)$ by characters of Z_G^{st}, and conjecturally this decomposition is only slightly coarser than the decomposition by L-packets. Thus, the stable center conjecture can be thought both as a supporting evidence and as a step in the proof of the local Langlands conjecture.

As follows from results of Bernstein and Moy-Prasad, the category $R(G)$ decomposes as a direct sum $R(G) = R(G)^0 \oplus R(G)^{>0}$, where $R(G)^0$ (resp. $R(G)^{>0}$) consists of those representations π, all of whose irreducible subquotients have depth zero (resp. positive depth). Therefore the Bernstein center Z_G decomposes as a direct sum of centers $Z_G = Z_G^0 \oplus Z_G^{>0}$. In particular, we have an embedding $Z_G^0 \hookrightarrow Z_G$, which identifies the unit element of Z_G^0 with the projector to the depth zero spectrum $z^0 \in Z_G$. Set $Z_G^{st,0} := Z_G^0 \cap Z_G^{st}$.

The depth zero stable center conjecture asserts that $Z_G^{st,0} \subset Z_G^0$ is a unital subalgebra. In particular, it predicts the stability of the projector $z^0 \in Z_G$.

The main goal of this work is to outline an approach to a proof of the depth zero stable center conjecture. As an illustration of our method, we prove an explicit formula for the Bernstein projector z^0, and deduce its stability. More precisely, we do it when G is a split semisimple simply connected group, and F is a local field of a positive but not very small characteristic.

Our approach. — Our strategy is to construct explicitly many elements z of $Z_G^0 \subset Z_G$, whose span is a subalgebra, and to prove that these elements are stable and generate all of $Z_G^{st,0}$. Here by "explicitly", we mean to describe both the invariant distribution ν_z on $G(F)$ and the function f_z on $\mathrm{Irr}(G)$.

To carry out our strategy, we construct first a categorical analog $\mathcal{Z}(LG)$ of the Bernstein center Z_G. Then we observe that a version of the Grothendieck "sheaf-function correspondence" associates to each Frobenius equivariant object $\mathcal{F} \in \mathcal{Z}(LG)$ an element of the Bernstein center $[\mathcal{F}] \in Z_G$. Thus, to construct elements of Z_G, it suffices to construct Frobenius-equivariant objects of $\mathcal{Z}(LG)$.

In order to construct elements of $\mathcal{Z}(LG)$, we construct first a categorical analog $\mathcal{Z}_{\mathbf{I}^+}(LG)$ of Z_G^0 and a categorical analog $\mathcal{A} : \mathcal{Z}_{\mathbf{I}^+}(LG) \to \mathcal{Z}(LG)$ of the embedding $Z_G^0 \hookrightarrow Z_G$. Then we apply $\mathcal{A}$ to monodromic analogs of Gaitsgory central sheaves.

Roughly speaking, we define $\mathcal{A}$ to be the composition of the averaging functor $\mathrm{Av}_{\mathrm{Fl}}$, where Fl is the affine flag variety of G, and the functor of "derived $\widetilde{W}$-skew-invariants", where $\widetilde{W}$ is the affine Weyl group of G. This construction is motivated by an analogous finite-dimensional result, proven in [**BKV1**]. However, in the affine case one has to overcome many technical difficulties.

Bernstein projector to the depth zero spectrum. — To illustrate our method, we provide a geometric construction of the Bernstein projector $z^0 \in Z_G$. More precisely, we construct a class $\langle A\rangle$ in the Grothendieck group version of $\mathcal{Z}(LG)$ and show that the corresponding element of Z_G is z^0. Then we show that the restriction $\nu_{z^0}|_{G^{rss}(F)}$ is locally constant and prove an explicit formula, which we now describe.

Let I^+ be the pro-unipotent radical of the Iwahori subgroup of $G(F)$, let μ^{I^+} be the Haar measure on $G(F)$ normalized by the condition that $\mu^{I^+}(I^+)=1$, and let $\phi_{z^0}\in C^\infty(G(F))$ be such that $\nu_{z^0}|_{G^{rss}(F)}=\phi_{z^0}\mu^{I^+}$.

For each $\gamma\in G^{rss}(F)$, we denote by Fl_γ be the corresponding affine Springer fiber. The affine Weyl group $\widetilde{W}$ acts on each homology group $H_i(\mathrm{Fl}_\gamma)=H^{-i}(\mathrm{Fl}_\gamma,\mathbb{D}_{\mathrm{Fl}_\gamma})$, where $\mathbb{D}_{\mathrm{Fl}_\gamma}$ is the dualizing sheaf. Consider the Tor-groups $\mathrm{Tor}_j^{\widetilde{W}}(H_i(\mathrm{Fl}_\gamma),\mathrm{sgn})$, where by sgn we denote the sign-character of $\widetilde{W}$. Each $\mathrm{Tor}_j^{\widetilde{W}}(H_i(\mathrm{Fl}_\gamma),\mathrm{sgn})$ is a finite-dimensional $\overline{\mathbb{Q}_l}$-vector space, equipped with an action of the Frobenius element. One of the main results in this paper is the following identity

$$\phi_{z^0}(\gamma)=\sum_{i,j}(-1)^{i+j}\,\mathrm{Tr}(\mathrm{Fr},\mathrm{Tor}_j^{\widetilde{W}}(H_i(\mathrm{Fl}_\gamma),\mathrm{sgn})). \tag{0.1}$$

Using formula (0.1) and a group version of a theorem of Yun [**Yun**], we show that $\nu_{z^0}|_{G^{rss}(F)}$ is stable. Note that the proof of Yun is global, while all the other arguments are purely local.

Remark. — Though ∞-categories are not needed for the construction of $\langle A\rangle$, we need them in order to prove the formula (0.1). Moreover, the structure of formula (0.1) indicates why the ∞-categories appears here. The shape of the formula suggests a possibility to write the right hand side of (0.1) as the trace of Frobenius on the "derived skew-coinvariants" $R\Gamma(\mathrm{Fl},\mathbb{D}_{\mathrm{Fl}_\gamma})_{\widetilde{W},\mathrm{sgn}}$. However, the functor of "derived skew-coinvariants" is defined as a homotopy colimit, and it can not be defined in the framework of derived categories. Therefore one has to pass to stable ∞-categories.

Plan of the paper. — In Section 1 we study derived categories of constructible sheaves on a certain class of ind-schemes and ind-stacks, which we call admissible. This class includes some infinite-dimensional ind-stacks, which are not algebraic. We also construct a certain geometric 2-category, whose ∞-version is used later.

In Section 2, we apply the formalism of Section 1 to the case of loop groups LG and related spaces in order to construct a categorical analog of the Hecke algebra.

Section 3 deals with the stable center conjecture. Namely, we formulate and discuss the stable center conjecture in subsection 3.1, categorify various objects from subsection 3.1 in subsections 3.2-3.3, and describe our (conjectural) approach to the depth zero stable center conjecture in subsections 3.4-3.5.

The results in subsections 3.2-3.3 are given without complete proofs, and details will appear in the forthcoming paper [**BKV2**]. To emphasise this fact, we write *"Theorem"* instead of *Theorem*.