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FOURIER TRANSFORM OF ALGEBRAIC MEASURES

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FOURIER TRANSFORM OF ALGEBRAIC MEASURES

by

Vladimir Drinfeld

To Gérard Laumon on his 60th birthday

Abstract. — These are notes of a talk based on the work [**AD**] joint with A. Aizenbud.

Let V be a finite-dimensional vector space over a local field F of characteristic 0. Let f be a function on V of the form $x \mapsto \psi(P(x))$, where P is a polynomial on V and ψ is a nontrivial additive character of F. Then it is clear that the Fourier transform $\mathrm{Four}(f)$ is well-defined as a distribution on V^*. Due to J.Bernstein, Hrushovski-Kazhdan, and Cluckers-Loeser, it is known that $\mathrm{Four}(f)$ is smooth on a non-empty Zariski-open conic subset of V^*. The goal of these notes is to sketch a proof of this result (and some related ones), which is very simple modulo resolution of singularities (the existing proofs use D-module theory in the Archimedean case and model theory in the non-Archimedean one).

***Résumé* (Transformation de Fourier de mesures algébriques).** — Ce sont les notes d'un exposé basé sur le travail [**AD**] commun avec A. Aizenbud.

Soit V un espace vectoriel de dimension finie sur un corps local F de caractéristique 0. Soit f une fonction sur V de la forme $x \mapsto \psi(P(x))$, où P est un polynome sur V et ψ est un caractère additif non trivial de F. Alors il est clair que la transformée de Fourier $\mathrm{Four}(f)$ est bien définie comme distribution sur V^*. D'après J.Bernstein, Hrushovski-Kazhdan et Cluckers-Loeser, il est connu que $\mathrm{Four}(f)$ est lisse sur un sous-ensemble ouvert de Zariski conique de V^*. Le but de ces notes est d'esquisser une démonstration de ce résultat (et de résultats liés), qui est très simple modulo la résolution des singularités (les preuves existantes utilisent la théorie des D-modules dans le cas archimédien et la théorie des modèles dans le cas non archimédien).

These are notes of a talk based on the work [**AD**] joint with A. Aizenbud. The results from [**AD**] are formulated in §§1–3, the proofs are sketched in §§4-5.

In Appendix A we discuss some "baby examples"; this material is not contained in [**AD**].

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Key words and phrases. — Wave front set, Fourier transform, distributions, oscillating integrals, resolution of singularities, local fields.

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I thank D. Kazhdan for drawing my attention to the questions considered in these notes. I also thank M. Kashiwara for communicating to me Example A.1 from Appendix A.

1. A theorem on Fourier transform

Let F be a local field of characteristic 0 (Archimedean or not). Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character.

Let V be a finite-dimensional vector space over F and $P : V \to F$ a polynomial. Then $\psi(P(x))$ is a smooth [(1)] $\mathbb{C}$-valued function on V.

Consider $\widehat{\psi(P(x))}$, *i.e.*, the Fourier transform of the function $\psi(P(x))$.

Note that in the naive sense the Fourier transform is not defined because $\psi(P(x))$ doesn't decay as $x \to \infty$, rather it oscillates. However, $\widehat{\psi(P(x))}$ *is well-defined as a distribution* [(2)] *on* V^*.

In the non-Archimedean case this is clear because the Fourier transform in the sense of distributions is well-defined for *any* generalized function, in particular, for any smooth function.

In the Archimedean case the Fourier transform is well-defined for generalized functions of *moderate growth*, and of course, $\psi(P(x))$ has moderate growth.

Theorem 1.1. — *There exists a Zariski-open $U \subset V^*$, $U \neq \varnothing$, such that the distribution $\widehat{\psi(P(x))}$ is smooth on U.*

If F is Archimedean Theorem 1.1 was proved by J. Bernstein [**Ber1**] using *D-module theory*. Moreover, he proved that $\widehat{\psi(P(x))}$ satisfies a holonomic system of linear p.d.e.'s with polynomial coefficients.

If F is non-Archimedean Theorem 1.1 was proved by Kazhdan-Hrushovski [**HK**] and Cluckers-Loeser [**CL**] using *model theory*. In both articles Theorem 1.1 appears as one of many corollaries of a general theory, and this general theory is quite different from D-module theory used by Bernstein.

The goal of these notes is to explain another proof of Theorem 1.1 and its refinements, namely the one from [**AD**]. It works equally well in the Archimedean and non-Archimedean case. Unlike the older proofs, it uses resolution of singularities [(3)].

1. In the non-Archimedean case "smooth" means "locally constant", in the non-Archimedean case the word "smooth" is understood literally.

2. Our conventions are as follows: a distribution on a manifold M is a generalized measure (*i.e.*, a linear functional on the space of smooth *functions* with compact support), while a generalized function on M is a linear functional on the space of smooth *measures* with compact support. If M is a vector space then sometimes (but not here) we do not distinguish functions from measures.

3. In my talk I said that a variant of "local uniformization" (see [**Za**], [**ILO**]) would suffice. But the argument that I had in mind contained a gap.

On the other hand, once you believe in resolution of singularities, the rest is an exercise in elementary analysis (with a bit of elementary symplectic geometry).

Here are some refinements of Theorem 1.1.

Refinement A. The open subset U can be chosen to be independent of ψ.
Refinement B. The open subset U can be chosen to be defined over the field K generated by the coefficients of P.
Refinement B′. The open subset U can be chosen to work for all embeddings of K into all possible local fields. (This makes sense by virtue of A.)

2. A theorem that implies Theorem 1.1

Let W be a finite-dimensional vector space over F. Let X be a smooth algebraic variety over F and $\varphi : X \to W$ a proper morphism. Let ω be a regular top differential form on X. Then we have a measure $|\omega|$ on $X(F)$. Set $\mu := \varphi_*|\omega|$ (note that φ_* is well-defined because φ is proper); μ is a measure on the vector space W, in particular, it is a distribution. Its Fourier transform, $\hat{\mu}$, is a well-defined [4] generalized function on W^*; it depends on the choice of ψ. The next theorem is an analog of Theorem 1.1 and its Refinements A,B,B′.

Theorem 2.1. — *There exists a non-empty Zariski-open $U \subset W^*$, independent of ψ, such that $\hat{\mu}$ is smooth on U. Moreover, if (X, φ, ω) is defined over a subfield $K \subset F$ then one can choose U to be defined over K and to have the required property for all embeddings of K into all possible local fields.*

Remark 2.2. — In Theorem 2.1 independence of U on ψ is equivalent to *stability of U under homotheties* of W. (This was not the case in the situation of Theorem 1.1 because ψ occurred there twice: in the definition of Fourier transform and in the expression $\psi(P(x))$.)

Let us show that Theorem 2.1 implies Theorem 1.1 and its refinements formulated at the end of §1. To prove this, apply Theorem 2.1 as follows. Set $W := V \oplus F = V \times F$, $X := V$. Define $\varphi : V \to V \times F$ by $\varphi(v) := (v, P(v))$. Take ω to be an invariant differential form on $V = X$.

Then the generalized function $\hat{\mu}$ on $W^* = V^* \times F$ is equal to the continuous map

$$F \longrightarrow \{\text{generalized functions on } V^*\}$$

that takes $\eta \in F$ to the Fourier transform of $\psi(\eta \cdot P(x))$ with respect to $x \in V$.

So Theorem 2.1 says that the Fourier transform of $\psi(\eta \cdot P(x))$ with respect to *both* x and η (which is *a priori* a *generalized* function on $V^* \times F$) is, in fact, *smooth* on some non-empty open subset $U \subset V^* \times F$, $U \neq \varnothing$, which can be chosen to be stable

4. In the Archimedean case one has to check that μ has moderate growth. This is not hard and well known.