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COHOMOLOGY OF LARGE SEMIPROJECTIVE HYPERKÄHLER VARIETIES

by

Tamás Hausel & Fernando Rodriguez Villegas

À Gérard Laumon à l'occasion de son 60éme anniversaire

***Abstract.* —** In this paper we survey geometric and arithmetic techniques to study the cohomology of semiprojective hyperkähler manifolds including toric hyperkähler varieties, Nakajima quiver varieties and moduli spaces of Higgs bundles on Riemann surfaces. The resulting formulae for their Poincaré polynomials are combinatorial and representation theoretical in nature. In particular we will look at their Betti numbers and will establish some results and state some expectations on their asymptotic shape.

***Résumé* (Cohomologie des variétés hyperkähleriennes semiprojectives grandes). —** Dans cet article, nous passons en revue les techniques géométriques et arithmétiques pour étudier la cohomologie des variétés hyperkählériennes semiprojectives, en particulier les variétés hyperkählériennes toriques, les variétés de carquois de Nakajima et les espaces de modules de fibrés de Higgs sur les surfaces de Riemann. Les formules obtenues pour leurs polynômes de Poincaré sont de nature combinatoire et liées à la théorie des représentations. En particulier, nous étudions leurs nombres de Betti et nous établissons des résultats et formulons quelques hypothèses sur leur comportement asymptotique.

At the conference "De la géométrie algébrique aux formes automorphes: une conférence en l'honneur de Gérard Laumon" the first author gave a talk, whose subject is well-documented in the survey paper [**Ha4**]. Here, instead, we will discuss techniques, both geometrical and arithmetic, for obtaining information on the cohomology of semiprojective hyperkähler varieties and we will report on some observations on the asymptotic behaviour of their Betti numbers in certain families of examples.

We call X a smooth quasi-projective variety with a $\mathbb{C}^\times$-action *semiprojective* when the fixed point set $X^{\mathbb{C}^\times}$ is projective and for every $x \in X$ and as $\lambda \in \mathbb{C}^\times$ tends to 0 the limit $\lim_{\lambda\to 0} \lambda x$ exists.

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***Key words and phrases.* —** Cohomology, Betti numbers, quiver variety, hyperkähler variety, Hard Lefschetz Theorem.

Varieties with these assumptions were originally studied by Simpson in [**Si2**, §11] and varieties with similar assumptions were studied by Nakajima in [**Na3**, §5.1]. The terminology semiprojective in this context appeared in [**HS**], which concerned semiprojective toric varieties and toric hyperkähler varieties. In particular, a large class of hyperkähler varieties, which arise as a hyperkähler quotient of a vector space by a gauge group, are semiprojective. These include Hilbert schemes of n-points on $\mathbb{C}^2$, Nakajima quiver varieties and moduli spaces of Higgs bundles on Riemann surfaces.

It turns out that despite their simple definition we can say quite a lot about the geometry and cohomology of semiprojective varieties. We can construct a Bialynicki-Birula stratification (§1.2), which in §1.3 will give a perfect Morse stratification in the sense of Atiyah-Bott. This way we will be able to deduce that the cohomology of a semiprojective variety is isomorphic with the cohomology of the fixed point set $X^{\mathbb{C}^\times}$ with some cohomological shifts. Also, the opposite Bialynicki-Birula stratification will stratify a projective subvariety $\mathcal{C} \subset X$ of the semiprojective variety, the so-called *core*, which turns out to be a deformation retract of X. This way we can deduce that the cohomology $H^*(X;\mathbb{C})$ is always pure. Furthermore, we can compactify $\overline{X} = X \coprod Z$ with a divisor Z, to get an orbifold $\overline{X}$. Finally in §1.4 we will look at a version of a weak form of the Hard Lefschetz theorem satisfied by semiprojective varieties.

We will also discuss arithmetic approach to obtain information on the cohomology of our hyperkähler varieties. It turns out that the algebraic symplectic quotient construction of our hyperkähler varieties will enable us to use a technique we call *arithmetic harmonic analysis* to count the points of our hyperkähler varieties over finite fields. With this technique we can effectively determine the Betti numbers of the toric hyperkähler varieties and Nakajima quiver varieties as well as formulate a conjectural expression for the Betti numbers of the moduli space of Higgs bundles.

To test the range in which the Weak Hard Lefschetz theorem of §1.4 might hold, we will look at the graph of Betti numbers for our varieties when their dimension is very large. The resulting pictures are fairly similar and we observe that asymptotically they seem to converge to the graph of some continuous functions. We will see, for example, the normal, Gumbel and Airy distributions emerging in the limit in our examples. We will conclude the paper with some proofs and heuristics towards establishing such facts.

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1. Semiprojective varieties

1.1. Definition and examples. — We start with the definition of a semiprojective variety, first considered in [**Si2**, Theorem 11.2].

Definition 1.1.1. — Let X be a complex quasi-projective algebraic variety with a $\mathbb{C}^\times$-action. We call X *semiprojective* when the following two assumptions hold:

1. The fixed point set $X^{\mathbb{C}^\times}$ is proper.
2. For every $x \in X$ the $\lim_{\lambda\to 0} \lambda x$ exists as $\lambda \in \mathbb{C}^\times$ tends to 0.

The second condition could be phrased more algebraically as follows: for every $x \in X$ we have an equivariant map $f : \mathbb{C} \to X$ such that $f(1) = x$ and $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication.

First example is a projective variety with a trivial (or any) $\mathbb{C}^\times$-action. For a large class of non-projective examples one can take the total space of a vector bundle on a projective variety, which together with the canonical $\mathbb{C}^\times$-action will become semiprojective.

A good source of examples arise by taking GIT quotients of linear group actions of reductive groups on vector spaces. Examples include the semiprojective toric varieties of [**HS**] (even though the definition of semiprojectiveness is different there, but equivalent with ours) and quiver varieties studied by Reineke [**Re1**].

1.1.1. Semiprojective hyperkähler varieties. — In this survey we are interested in semiprojective hyperkähler varieties. Examples arise by taking the algebraic symplectic quotient of a complex symplectic vector space $\mathbb{M}$ by a symplectic linear action of a reductive group $\overline{\rho} : \mathrm{G} \to \mathrm{Sp}(\mathbb{M})$. In practice $\mathbb{M} = \mathbb{V} \times \mathbb{V}^*$ and $\overline{\rho}$ arises as the doubling of a representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$. If $\mathfrak{g}$ denotes the Lie algebra of G, we have the derivative of ρ as $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$. This gives us the moment map

$$\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*, \tag{1.1.1}$$

at $x \in \mathfrak{g}$ by the formula

$$\langle \mu(v, w), x\rangle = \langle \varrho(x)v, w\rangle. \tag{1.1.2}$$

By construction μ is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$. Taking a character $\sigma \in \mathrm{Hom}(\mathrm{G}, \mathbb{C}^\times)$ will yield the GIT quotient $\mathcal{M}^\rho_\sigma := \mu^{-1}(0)/\!/_\sigma \mathrm{G}$ using the linearization induced by σ. Sometimes σ can be chosen generically so that $\mathcal{M}^\rho_\sigma$ becomes non-singular (and by construction) quasi-projective. We assume this henceforth. By construction of the GIT quotient we have the proper affinization map

$$\mathcal{M}^\rho_\sigma \longrightarrow \mathcal{M}^\rho_0 \tag{1.1.3}$$

to the affine GIT quotient $\mathcal{M}^\rho_0 = \mu^{-1}(0)/\!/\mathrm{G}$.

The $\mathbb{C}^\times$-action on $\mathbb{M}$ given by dilation will commute with the linear action of G on it so that the moment map (1.1.1) will be equivariant with respect to this and the weight 2 action of $\mathbb{C}^\times$ on $\mathfrak{g}^*$. This will induce a $\mathbb{C}^\times$-action on $\mathcal{M}^\rho_\sigma$, such that on the affine GIT quotient $\mathcal{M}^\rho_\sigma$ it will have a single fixed point corresponding to the origin