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A CONJECTURE OF HAUSEL ON THE MODULI SPACE OF HIGGS BUNDLES ON A CURVE

by

Jochen Heinloth

To Gérard Laumon

Abstract. — In this survey, we review some conjectures on the cohomology of the moduli space of Higgs bundles on a curve and explain joint work with O. Garcia-Prada and A. Schmitt [**10, 9**] resulting in a recursive algorithm to determine the cohomology of moduli spaces of semi-stable Higgs bundles on a curve (in the coprime situation). This method allows us to confirm a conjecture of T. Hausel who predicted a formula for the y-genus of the moduli space.

***Résumé* (Une conjecture de Hausel sur l'espace des fibrés de Higgs sur une courbe).** — Nous passons en revue des conjectures concernant la cohomologie des espaces de fibrés de Higgs sur une courbe et expliquons un algorithme de calcul de la cohomologie, trouvé en collaboration avec O. Garcia-Prada et A. Schmitt [**10, 9**]. Cet algorithme permet de confirmer la conjecture de T. Hausel sur le genre y de l'espace des fibrés de Higgs.

1. Introduction

The moduli space of Higgs bundles on a projective curve C was introduced by Hitchin [**17**], who discovered several remarkable geometric properties of this space. Roughly the space can be thought of as the cotangent space to the moduli space of vector bundles on a given curve and Hitchin showed that this space carries the structure of an integrable system. If C is defined over the complex numbers, the moduli space admits a natural family of complex structures, *e.g.*, it turns out to be diffeomorphic to the so called character variety, *i.e.*, the space of representations of the fundamental group of the curve.

Much later, Ngô [**22**] exhibited another use of this moduli space by observing, that for curves defined over finite fields, the adelic description of the stack of Higgs bundles on the curve is closely related to spaces occurring in the study of the trace formula.

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Despite this wealth of structure, the cohomology of the moduli space has not yet been determined. For bundles of small rank ($n = 2, 3$) the answer has been known for a long time by work of Hitchin and Gothen, but from these results it did not seem possible to come up with a general prediction. It was therefore a surprise, when T. Hausel and F. Rodriguez-Villegas managed to formulate a conjecture, predicting the additive structure of the cohomology.

In this note we want to survey joint work with O. Garcia-Prada and A. Schmitt ([**10, 9**]), in which we found an algorithm for the computation of the cohomology of the moduli space of Higgs bundles on a curve (if rank and degree are coprime) and applied this to prove a specialization of the conjecture by Hausel and Rodriguez-Villegas.

Let me briefly summarize the structure of the article. In Section 2 we recall the basic definitions and results on moduli spaces of Higgs bundles. Since Hausel's conjectures originated from insights on mirror symmetry, we take the opportunity to relate one of these conjectures to Ngô's support theorem. In Section 3 we recall the conjecture of Hausel and Rodriguez-Villegas giving a formula for the cohomology of the moduli space of Higgs bundles. Here we use the reformulation introduced by S. Mozgovoy in terms of the Grothendieck ring of varieties. We end the section by explaining how Hausel used this conjecture to deduce an explicit conjecture for the Hirzebruch y-genus of the moduli space. In Section 4 we then review how we obtained the aforementioned algorithm and deduce Hausel's conjecture on the y-genus.

2. Some properties of the moduli space of Higgs bundles and its cohomology

In this section we will recall the definition and some of the basic properties of the moduli space of Higgs bundles on a curve. These definitions and most of the results are due to Hitchin [**17**]. We then deduce some basic results on the cohomology of the moduli space. Almost all of these results seem to be well known, but we use the opportunity to relate one of Hausel's conjectures to Ngô's support theorem [**23**].

We will denote by C a fixed smooth projective, geometrically connected, algebraic curve of genus g defined over some field k. For our application we will only need to consider $k = \mathbb{C}$, but sometimes the possibility to choose k to be a finite field was useful for us. We will often abbreviate cohomology of a variety X by $H^*(X)$. If $k = \mathbb{C}$ we want to understand that $H^*(X) = H^*(X, \mathbb{Q})$ is singular cohomology with rational coefficients, equipped with its natural mixed Hodge structure on $H^*(X) \otimes_{\mathbb{Q}} \mathbb{C}$. If k is a general field, we abbreviate $H^*(X) = H^*_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ for some prime number ℓ prime to the characteristic of k. If $\mathcal{F}$ is a coherent sheaf on X we will denote by $H^*(X, \mathcal{F})$ the usual cohomology groups of coherent sheaves.

A *Higgs bundle* on C is a pair $(\mathcal{E}, \theta \colon \mathcal{E} \to \mathcal{E} \otimes \Omega_C)$, where $\mathcal{E}$ is a vector bundle on C, θ is an $\mathcal{O}_C$-linear map and Ω_C is the sheaf of differentials on C. To motivate this definition, recall that deformations of a vector bundle $\mathcal{E}$ on C are parametrized by

$H^1(C, \mathrm{End}(\mathcal{E}))$ and by Serre duality, this space is dual to $H^0(C, \mathrm{End}(\mathcal{E}) \otimes \Omega_C)$, so one can think of a Higgs bundle as a point in the cotangent space to the moduli stack of vector bundles. However, as we will see below, one has to be careful when using this interpretation.

We denote by $\mathcal{M}_{n,d}$ the moduli stack of Higgs bundles of rank n and degree d on C, *i.e.*, the stack classifying pairs $(\mathcal{E}, \theta\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_C)$ as above such that $\deg(\mathcal{E}) = d$, $\mathrm{rk}(\mathcal{E}) = n$.

As an immediate warning let us mention that this stack is very big. Already the stack of vector bundles on a curve is only locally of finite type, but for any fixed degree and any $N \geqslant 0$ the stack of vector bundles has an open substack of finite type, such that the complement has codimension $> N$ (see *e.g.*, [**4**]). This property usually fails for the stack of Higgs bundles (even for $C = \mathbb{P}^1$ and $n = 2$).

To see where this problem comes from, let us recall that (by a result of Biswas and Ramanan [**6**]) infinitesimal deformations of a Higgs bundle $(\mathcal{E}, \theta)$ are described by the cohomology of the complex

$$C_\bullet(\mathcal{E}, \theta) := (\mathrm{End}(\mathcal{E}) \longrightarrow \mathrm{End}(\mathcal{E}) \otimes \Omega_C).$$

The tangent space of the stack at $(\mathcal{E}, \theta)$ is the quotient stack

$$[H^1(C, C_\bullet(\mathcal{E}, \theta))/H^0(C, C_\bullet(\mathcal{E}, \theta))].$$

Now Serre duality implies that the dimension of H^0 of the complex is equal to the dimension of H^2, so whenever the Higgs bundle admits non-trivial automorphisms the dimension of this tangent space will increase. In particular, the stack of Higgs bundles will be singular in general.

Moreover, recall that in the deformation theory of vector bundles whenever the dimension of $H^1(C, \mathrm{End}(\mathcal{E}))$ increases, the dimension of the automorphism group of the bundle also increases to cancel this variation. For Higgs bundles, we see that automorphisms of Higgs bundles only compensate half of the variation of the dimension of H^1, which causes the problem that $\mathcal{M}_{n,d}$ is much bigger than expected.

Since stability of Higgs bundles allows us to forget about these problems, we will need to recall this notion. For a vector bundle $\mathcal{E}$ on C the slope of $\mathcal{E}$ is defined as $\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$. A Higgs bundle $(\mathcal{E}, \theta)$ is called *semistable* if for all subsheaves $\mathcal{F} \subset \mathcal{E}$ with $\theta|_\mathcal{F}\colon \mathcal{F} \to \mathcal{F} \otimes \Omega_C$ we have $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$. A Higgs-bundle is called *stable* if this last inequality is a strict inequality for all proper $(\mathcal{F}, \theta_{|\mathcal{F}}) \subsetneq (\mathcal{E}, \theta)$. Note that as in the case of vector bundles the notions of semistability and stability coincide, whenever $\mathrm{rk}(\mathcal{E})$ and $\deg(\mathcal{E})$ are coprime.

(Semi-)stability is an open condition on families of Higgs bundles, so we can consider the substack of semistable Higgs bundles $\mathcal{M}^{ss}_{n,d} \subset \mathcal{M}_{n,d}$. As in the case of vector bundles, stable Higgs bundles only admit scalar endomorphisms, so that for a stable Higgs bundle we have $\dim H^2(C, \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C) = 1$. Since the Euler characteristic of the cohomology $H^*(C, \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C)$ is

$$n^2(1-g) - (n^2(2g-2) + n^2(1-g)) = -2n^2(g-1).$$