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ELEMENTARY ABELIAN ℓ -GROUPS AND
mod ℓ EQUIVARIANT ÉTALE COHOMOLOGY ALGEBRAS

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ELEMENTARY ABELIAN ℓ -GROUPS AND MOD ℓ EQUIVARIANT ÉTALE COHOMOLOGY ALGEBRAS

by

Luc Illusie

À Gérard, avec affection et admiration

Abstract. — This article is a report on joint work with W. Zheng [8]. We give an overview of the main results and sketch their proofs. They mainly consist in variants and generalizations, in the framework of étale cohomology, of theorems of Quillen [14, 15]. Here is an example: if k is an algebraically closed field and ℓ is a prime number different from the characteristic of k , X a separated k -algebraic space of finite type, equipped with an action of a k -algebraic group G , the equivariant étale cohomology algebra $H^*([X/G], \mathbf{F}_\ell)$ is finitely generated and is F -isomorphic to a finite projective limit of algebras of the form $H^*(A, \mathbf{F}_\ell)$ for A an ℓ -elementary abelian subgroup of G fixing a point in X .

Résumé (ℓ -groupes abéliens élémentaires et algèbres de cohomologie étale équivariante mod ℓ). — Cet article est un rapport sur un travail en commun avec W. Zheng [8]. Nous donnons un aperçu des principaux résultats et des indications sur leurs démonstrations. Il s'agit, pour l'essentiel, de variantes et généralisations, en cohomologie étale, de théorèmes de Quillen [14, 15]. En voici un exemple : si k est un corps algébriquement clos et ℓ un nombre premier différent de la caractéristique de k , X un k -espace algébrique séparé et de type fini, muni d'une action d'un k -groupe algébrique G , l'algèbre de cohomologie étale équivariante $H^*([X/G], \mathbf{F}_\ell)$ est de type fini, et est F -isomorphe à une limite projective finie d'algèbres de la forme $H^*(A, \mathbf{F}_\ell)$ pour A un sous-groupe abélien ℓ -élémentaire de G fixant un point de X .

This is a report on joint work with W. Zheng [8]. It grew out of questions that Serre asked me about traces for finite group actions. These questions were the subject of the previous joint papers [6] and [7]. They led us to consider more generally actions of algebraic groups and revisit, in the context of mod ℓ étale cohomology, a theory of equivariant cohomology developed in the early 70's by Quillen for actions of compact Lie groups on topological spaces ([14, 15]).

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1. Finite ℓ -group actions, fixed point sets and localizations

Let k be an algebraically closed field of characteristic p and ℓ a prime number $\neq p$. Let X be a separated k -scheme of finite type, acted on by a finite ℓ -group G . Serre ([18, §7.2]) observed that we have the following identity

$$(1.1) \quad \chi(X) \equiv \chi(X^G) \bmod \ell.$$

Here X^G is the fixed point set of G , and $\chi = \chi(-, \mathbf{Q}_\ell) = \sum (-1)^i \dim H^i(-, \mathbf{Q}_\ell)$ denotes an Euler-Poincaré ℓ -adic characteristic. It has been known since the early sixties that this integer does not depend on ℓ , as follows from Grothendieck's cohomological formula for the zeta function of a variety over a finite field. Recall also that, by a theorem of Laumon [11], $\chi = \chi_c := \sum (-1)^i \dim H_c^i(-, \mathbf{Q}_\ell)$.

The proof of (1.1) is immediate: by dévissage one reduces to the case where $G = \mathbf{Z}/\ell\mathbf{Z}$; in this case, as $|G| = \ell \neq p$ and G acts freely on $X - X^G$, by a theorem of Deligne (cf. [6, §4.3]) we have $\chi_c(X - X^G) = \ell \chi_c((X - X^G)/G)$, hence $\chi_c(X) = \chi_c(X^G) + \chi_c(X - X^G)$, and (1.1) follows from Laumon's result. When $G = \mathbf{Z}/\ell\mathbf{Z}$, for $g \in G$ we have a more precise result:

$$(1.2) \quad \mathrm{Tr}(g, H_c^*(X, \mathbf{Q}_\ell)) = \chi(X^G) + \chi((X - X^G)/G) \mathrm{Reg}_G(g),$$

where $\mathrm{Tr}(g, H_c^*) := \sum (-1)^i \mathrm{Tr}(g, H_c^i)$ and Reg_G denotes the character of the regular representation of G . In fact ([6, (2.3)]) $\mathrm{Tr}(g, H_c^*) = \mathrm{Tr}(g, H^*)$ (an equivariant form of Laumon's theorem).

In particular, if ℓ does not divide $\chi(X)$, then X^G is not empty. This is the case, for example, if X is the standard affine space \mathbf{A}_k^n of dimension n over k , as (1.1) implies $\chi(X^G) \equiv 1 \bmod \ell$. Serre ([18, §1.2]) remarks that in this case one can show $X^G \neq \emptyset$ in a much more elementary way: reduce to the case where k is the algebraic closure of a finite field $k_0 = \mathbf{F}_q$ and the action of G on $X = \mathbf{A}_{k_0}^n$ comes from an action of G on $X_0 = \mathbf{A}_{k_0}^n$. Then we have the stronger property $X_0(k_0)^G \neq \emptyset$, as $|X_0(k_0)| = q^n$ and ℓ divides the cardinality of any non trivial orbit. Given a field K and an action of a finite ℓ -group G on \mathbf{A}_K^n , Serre ([18, loc. cit.]) asks whether $\mathbf{A}_K^n(K)^G$ is not empty. This is the case for $n \leq 2$ (elementary for $n = 1$, by Esnault-Nicaise ([5, §5.12]) for $n = 2$). The answer is unknown for $n = 3$, $K = \mathbf{Q}$, $|G| = 2$. In the positive direction, in addition to the case where K is finite, Esnault-Nicaise ([5, §5.17]) prove that the answer is yes if K is a henselian discrete valuation field of characteristic zero whose residue field is of characteristic $\neq \ell$, and which is either algebraically closed or finite of cardinality q with $\ell \mid q - 1$. In the case $K = k$, Smith's theory gives more than the existence of a fixed point. Indeed we have:

Theorem 1.3 ([18, §7.9], [6, §7.3, §7.8]). — *Let X be an algebraic space separated and of finite type over k endowed with an action of a finite ℓ -group G . Then, if X is mod ℓ acyclic, so is X^G .*

Here, we say that Y/k is mod ℓ acyclic if $H^*(Y, \mathbf{F}_\ell) = H^0(Y, \mathbf{F}_\ell) = \mathbf{F}_\ell$. It is shown in loc. cit. that the conclusion of 1.3 still holds if the assumption $\ell \neq p$ made at the beginning of this section is dropped.

Sketch of proof of 1.3. — As in the proof of (1.1) we may assume by dévissage that $G = \mathbf{Z}/\ell\mathbf{Z}$. In this case, Serre's proof exploits the action of the algebra $\mathbf{F}_\ell[G]$ on $\pi_*(\mathbf{Z}/\ell\mathbf{Z})$, where $\pi : X \rightarrow X/G$ is the projection. The proof given in [6], which uses equivariant cohomology, is close in spirit to that of Borel [2] in the topological case. Let us first give a general definition.

For an algebraic space Y separated and of finite type over k endowed with an action of a finite group G , $R\Gamma(Y, \mathbf{F}_\ell)$ is an object of $D^+(\mathbf{F}_\ell[G])$. The equivariant cohomology complex of Y is defined as

$$(1.3.1) \quad R\Gamma_G(Y, \mathbf{F}_\ell) := R\Gamma(G, R\Gamma(Y, \mathbf{F}_\ell)),$$

which we will abbreviate here to $R\Gamma_G(Y)$. It has a natural multiplicative structure, and $H_G^*(Y) = H^*R\Gamma_G(Y)$ is a graded algebra over the graded \mathbf{F}_ℓ -algebra $H_G^* = H^*(G, \mathbf{F}_\ell)$. For $G = \mathbf{Z}/\ell\mathbf{Z}$, we have

$$(1.3.2) \quad H_{\mathbf{Z}/\ell\mathbf{Z}}^* = \begin{cases} \mathbf{F}_\ell[x] & \text{if } \ell = 2 \\ \mathbf{F}_\ell[x]/(x^2) \otimes \mathbf{F}_\ell[y] & \text{if } \ell > 2, \end{cases}$$

where x is the tautological generator of $H_{\mathbf{Z}/\ell\mathbf{Z}}^1$, and, for $\ell > 2$, $y = \beta x$, where $\beta : H_{\mathbf{Z}/\ell\mathbf{Z}}^1 \xrightarrow{\sim} H_{\mathbf{Z}/\ell\mathbf{Z}}^2$ is the Bockstein operator (associated with the exact sequence $0 \rightarrow \mathbf{F}_\ell \rightarrow \mathbf{Z}/\ell^2\mathbf{Z} \rightarrow \mathbf{F}_\ell \rightarrow 0$).

Coming back to the proof of 1.3, the key point is that (for $G = \mathbf{Z}/\ell\mathbf{Z}$) the restriction map

$$(1.3.3) \quad H_G^*(X) \longrightarrow H_G^*(X^G) = H_G^* \otimes H^*(X^G),$$

which is a map of graded H_G^* -modules, becomes an isomorphism after inverting $\beta x \in H_{\mathbf{Z}/\ell\mathbf{Z}}^2$. Indeed, the assumption that X is mod ℓ acyclic implies that $H_G^*(X) = H_G^*$, hence $H^*(X^G)$ has to be of rank one over \mathbf{F}_ℓ . The assertion about (1.3.3) follows from the fact that $H_G^*(X, j_! \mathbf{F}_\ell)$ is of bounded degree, where $j : X - X^G \hookrightarrow X$ is the inclusion, as X/G is of finite ℓ -cohomological dimension.

The above key point is similar to various *localization formulas* considered by Quillen, Atiyah-Segal, Goresky-Kottwitz-MacPherson. For actions of elementary abelian ℓ -groups⁽¹⁾ we have the following result, which is an analogue of Quillen's theorem ([14, §4.2]):

Theorem 1.4 ([6, §8.3]). — *Let X be an algebraic space separated and of finite type over k endowed with an action of an elementary abelian ℓ -group G of rank r , and let*

$$e := \prod_{\xi \in H_G^1 - \{0\}} \beta\xi \in H_G^{2(\ell^r - 1)},$$

1. An elementary abelian ℓ -group is a group G isomorphic to the direct product of a finite number r of cyclic groups of order ℓ . The integer r is called the rank of G .