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UNIPOTENT ALMOST CHARACTERS OF SIMPLE p-ADIC GROUPS

by

George Lusztig

To Gérard Laumon, on his 60th birthday

Abstract. — Let G be a simple adjoint group and let $K = k((\epsilon))$ where k is an algebraic closure of a finite field $\mathbf{F}_q$. In this paper we define some geometric objects on $G(K)$ which are similar to the (cohomology sheaves of the) unipotent character sheaves of $G(k)$. Using these geometric objects we define the unipotent almost characters of $G(K_0)$ where $K_0 = \mathbf{F}_q((\epsilon))$ and state some conjectures relating them to the characters of unipotent representations of $G(K_0)$.

***Résumé* (Presque caractères unipotents des groupes simples p-adiques).** — Soit G un groupe simple adjoint et soit $K = k((\epsilon))$ où k est une clôture algébrique d'un corps fini $\mathbf{F}_q$. Dans cet article nous définissons certains objets géométriques sur $G(K)$ qui sont similaires aux (faisceaux de cohomologie des) faisceaux-caractères unipotents de $G(k)$. En utilisant ces objets géométriques nous définissons les presque caractères unipotents de $G(K_0)$ où $K_0 = \mathbf{F}_q((\epsilon))$ et nous donnons quelques conjectures les reliant avec les caractères des représentations unipotentes de $G(K_0)$.

0. Introduction

0.1. Let G be a simple adjoint algebraic group defined and split over the finite field $\mathbf{F}_q$. Let $K_0 = \mathbf{F}_q((\epsilon))$, $K = \bar{\mathbf{F}}_q((\epsilon))$. We are interested in the characters of the standard representations of $G(K_0)$ corresponding to the (irreducible) unipotent representations ([**L6**]) of $G(K_0)$, restricted to the set $G(K_0)_{rsc} = G(K)_{rsc} \cap G(K_0)$ where $G(K)_{rsc}$ is the intersection of the set $G(K)_{rs}$ of regular semisimple elements in $G(K)$ with the set $G(K)_c$ of compact elements in $G(K)$ (that is, elements which normalize some Iwahori subgroup of $G(K)$); we call these restrictions the *unipotent*

Key words and phrases. — Character sheaf, almost character, unipotent representation, p-adic group.

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characters of $G(K_0)$. We hope that the unipotent characters (or some small linear combination of them) have a geometric meaning in the same way as the characters of (irreducible) unipotent representations of $G(\mathbf{F}_q)$ can be expressed in terms of character sheaves on G. Thus we are seeking some geometric objects on $G(K)_c$ on which the Frobenius map acts and from which the unipotent characters can be recovered.

In this paper we define a collection of class functions on $G(K_0)_{rsc}$ which we call *unipotent almost characters*. (These class functions can conceivably take the value ∞ at some points but we conjecture that the set of such points is empty; in the rest of this introduction we assume that this conjecture holds.) We expect that the unipotent almost characters are in the same relation with the unipotent characters as the objects with the same names associated to $G(\mathbf{F}_q)$. In particular we expect that the unipotent characters generate same subspace of the vector space of class functions on $\mathcal{V}_0$ as the unipotent almost characters. (A refinement of this is stated as a conjecture in 3.11 (a).) Moreover we expect that each unipotent almost character can be expressed as a linear combination of a small number of unipotent characters, just like for $G(\mathbf{F}_q)$.

Our definition of unipotent almost characters is similar to one of the two definitions of the analogous functions for $G(\mathbf{F}_q)$ (which was in terms of character sheaves on $G(\bar{\mathbf{F}}_q)$). They are associated to some new geometric objects on $G(K)_c$ which can be thought of as character sheaves on $G(K)_c$ (or rather, cohomology sheaves of character sheaves) and are defined even when $\mathbf{k}$ is replaced by any algebraically closed field such as $\mathbf{C}$ (in which case $G(K)$ becomes $G(\mathbf{C}(\epsilon))$).

The definition of these new geometric objects combines three ingredients:

(i) A generalization of the construction [**L8**] of an affine Weyl group action on the homology of the variety of Iwahori subgroups (see [**KL**]) of $G(K)$ containing a given element of $G(K)_c$.

(ii) A construction of co-standard representations of an affine Weyl group in the framework of the generalized Springer correspondence [**L3**].

(iii) A matching of the affine Weyl groups appearing in (i) and (ii).

Now (i) (which is discussed in §3) is based on some preliminaries on unipotent character sheaves on disconnected groups given in §1. It uses geometry (such as perverse sheaves) arising from $G(K)$. On the other hand, (ii) (which is discussed in §2) is a variant of the geometric construction of representations of graded affine Hecke algebras given in [**L4**].

It should be pointed out that the notion of co-standard module which appears in (ii) is not intrinsic to the affine Weyl group, but it is associated to an affine Weyl group viewed as the limit as $q \to 1$ of an affine Hecke algebras with possibly unequal parameters q^{m_i}. While the irreducible representations of the affine Weyl group in (ii) have an elementary definition (in terms of representations of finite Weyl groups), our definition of the co-standard representations is in terms of geometry (such as perverse sheaves) arising from the group of type dual to that of G.

Since the affine Weyl groups in (i) and (ii) appear in totally different worlds (one from G, the other from the dual group of G) the fact that they match is a miracle (which has already been exploited in [**L6**]).

After these new geometric objects are defined, the unipotent almost characters are defined in terms of them by taking traces of the Frobenius map.

In §4 we give some supporting evidence, based mostly on [**KmL2**], for the conjectures in this paper.

We expect that a similar picture exists with unipotent representations replaced by representations of depth zero. In §5 we discuss a possible generalization to p-adic groups in unequal characteristic.

0.2. If Γ is a group then $\mathcal{Z}_\Gamma$ is the centre of Γ. If H is a subgroup of Γ, then $N_\Gamma H$ is the normalizer of H in Γ. If $g \in \Gamma$ then $Z_\Gamma(g)$ is the centralizer of g in Γ. We denote by $\mathrm{Irr}\Gamma$ a set of representatives for the isomorphism classes of irreducible finite dimensional representations of Γ (over $\mathbf{C}$). We fix an algebraically closed field $\mathbf{k}$ and a prime number l invertible in $\mathbf{k}$. Let $\bar{\mathbf{Q}}_l$ be an algebraic closure of the field of l-adic numbers. For an algebraic variety X over $\mathbf{k}$ let $\mathcal{D}(X)$ be the bounded derived category of constructible $\bar{\mathbf{Q}}_l$ sheaves on X. By "local system" on X we usually mean a $\bar{\mathbf{Q}}_l$-local system. (An exception is in §2 where varieties and local systems are over $\mathbf{C}$.) If $\mathcal{E}$ is a local system on X and $i \in \mathbf{N}$ we set $H_{-i}(X, \mathcal{E}) = (H^i_c(X, \mathcal{E}^\dagger))^\dagger$; generally, $()^\dagger$ denotes the dual of a vector space or of a local system. If $x \in X$, $\mathcal{E}_x$ denotes the stalk of $\mathcal{E}$ at x. If H is a linear algebraic group over $\mathbf{k}$, let H^0 be the identity component of H. If H acts on X let $\mathcal{D}_H(X)$ be the corresponding equivariant derived category.

For a finite set S let $|S|$ be the cardinal of S.

In §3 we assume that $\bar{\mathbf{Q}}_l, \mathbf{C}$ are identified as fields.

1. Preliminaries on character sheaves on disconnected groups

1.1. Let G be an affine algebraic group over $\mathbf{k}$ such that G^0 is reductive. For any subgroup H of G we write NH instead of $N_G H$. The set of subgroups of G^0 containing a fixed Borel subgroup of G^0 is of the form $\{B_J; J \subset I\}$ (I is a finite indexing set) where for $J \subset I, J' \subset I$ we have $B_J \subset B_{J'}$ if and only if $J \subset J'$. In particular $B_\varnothing$ is a Borel subgroup. Let W be a (finite) indexing set for the set of $(B_\varnothing, B_\varnothing)$ double cosets in G. For $w \in W$ let O_w be the corresponding double coset. Let $W' = \{w \in W; O_w \subset G^0\}$ and let $\Xi = \{w \in W; O_w \subset NB_\varnothing\}$. If $i \in I$ then $B_{\{i\}} - B_\varnothing = O_w$ for a well defined $w \in W'$; we set $w = s_i$. There is a unique group structure on W such that the following holds: if $w, w' \in W$ are such that $O_w O_{w'}$ is of the form $O_{w''}$ for some $w'' \in W$ then $ww' = w''$; if $i \in I$ then $s_i^2 = 1$. Now W' is the subgroup of W generated by $\{s_i; i \in I\}$; it is a Coxeter group on these generators. Also Ξ is a subgroup of W such that $W = \Xi W' = W'\Xi$, $\Xi \cap W' = \{1\}$. If $\xi \in \Xi$ and $i \in I$ then $\xi s_i \xi^{-1} = s_{\xi(i)}$ for a unique $\xi(i) \in I$; moreover $\xi : i \mapsto \xi(i)$ is an action of Ξ on I. For $J \subset I$, let W_J be the subgroup of W' generated by $\{s_i; i \in J\}$ and let $\Xi_J = \Xi \cap N_W W_J$; let w_0^J be the longest element of the finite Coxeter group W_J.