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Analytic Continuation of Overconvergent Hilbert Modular Forms

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ANALYTIC CONTINUATION OF OVERCONVERGENT HILBERT MODULAR FORMS

by

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Abstract. — In these notes, we explain some recent progress on analytic continuation of overconvergent p-adic Hilbert modular forms and applications to classicality as well as the strong Artin conjecture. We will begin with the classical case of elliptic modular forms to explain the basic ideas and hint at what new ideas are needed in the general case. We then move on to the case of Hilbert modular forms where the prime p is assumed unramified in the relevant totally real field.

***Résumé* (Prolongement analytisque des formes modulaires surconvergentes de Hilbert)**
Dans ces notes, nous expliquons des progrès récents relatifs au prolongement analytique de formes modulaires surconvergentes p-adiques de Hilbert. Nous donnons des applications aux problèmes de classicité de telles formes ainsi qu'à la conjecture d'Artin forte. Nous commençons par le cas usuel des formes modulaires elliptiques pour dégager les idées simples et souligner les généralisations requises. Nous nous focalisons ensuite sur le cas des formes de Hilbert lorsque le nombre premier p est non ramifié dans le corps totalement réel.

The main theorems of this paper will be Theorems 3.4.4, 4.0.1 and 5.1.1 for which we defer to the rest of the text.

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1. The classical case

1.1. — In [**5**], Buzzard and Taylor proved the modularity of a certain kind of a Galois representation ρ by first showing that ρ arises from an overconvergent modular form f, and then proving that f is indeed a classical modular form. In this work (and the subsequent generalization by Buzzard [**4**]), the demonstration of the classicality of f was carried out through analytic continuation of f from its original domain of definition (which is an admissible open region in the rigid analytic modular curve) to the entire modular curve. This implies classicality since by the rigid analytic GAGA, any global analytic section of a line bundle over the analytification of a smooth projective variety is, indeed, algebraic.

Earlier, in **[6]**, Coleman had proved a criterion for classicality of p-adic overconvergent modular forms in terms of slope, i.e., the p-adic valuation of the eigenvalue of the U_p Hecke operator.

***Theorem 1.1.1* (Coleman).** — *Any overconvergent modular form f of weight k and slope less than $k-1$ is classical.*

Coleman's proof involved calculations with the cohomology of modular curves. We could, however, ask whether this result could be proven by invoking the above principle of analytic continuation. In other words, given the slope condition, could we analytically continue f from its domain of definition to the entire modular curve? In **[11]**, we showed that this is possible and involves the construction of a series whose convergence is guaranteed by the given slope condition. In this section, we will explain the proof in **[11]** by dissecting the method to see what is essential for the application of the method in more general cases. In doing so, we will introduce an idea of Pilloni which allow for a less explicit and, hence, more general approach.

1.2. The proof of Coleman's theorem via analytic continuation [11]

Let p be a prime number, and $N \geqslant 4$ an integer. In this chapter only, we let Y denote the completed modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$ defined over $\mathbb{Q}_p$. Its noncuspidal locus classifies the data $(\underline{E}, H)$ over $\mathbb{Q}_p$-schemes, where $\underline{E}$ is an elliptic curve with $\Gamma_1(N)$-level structure, and H a finite flat subgroup scheme of E of order p. Let ω be the usual sheaf on Y whose sections are invariant differentials on the universal family of (generalized) elliptic curves on Y. Modular forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $k \in \mathbb{Z}$ are elements of $H^0(Y, \omega^k)$. We let Y^{an} denote the p-adic rigid analytification of Y, and continue to denote the analytification of ω by ω. Let $Y_{\mathbb{Z}}$ denote the semistable integral model of Y defined using an integral version of the same moduli problem.

Let $Y^{\mathrm{an},0}$ denote the modular curve whose noncuspidal locus classifies all $(\underline{E}, H, D)$ such that $(\underline{E}, H) \neq (\underline{E}, D)$ and both are classified by Y^{an}. There are two morphisms $\pi_1, \pi_2 : Y^{\mathrm{an},0} \to Y^{\mathrm{an}}$ sending $(\underline{E}, H, D)$ to $(\underline{E}, H)$ and $(\underline{E}/D, \bar{H})$, respectively, where $\bar{H}$ denotes the image of H in $\underline{E}/D$.

To define rigid analytic regions inside Y^{an}, we need to recall the notion of degree of a finite flat group scheme over a finite extension of $\mathbb{Q}_p$ and some of its properties.

The degree of a finite flat group scheme. — We define the notion of degree and record some properties that we will use later. This useful notion was defined by Illusie and others, and has been more recently studied by Fargues in **[8]**. Let ν_p denote the p-adic valuation on $\mathbb{C}_p$ (the completion of an algebraic closure of $\mathbb{Q}_p$) such that $\nu_p(p) = 1$.

Definition 1.2.1. — Let $\mathcal{O}_K$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$. If G is finite flat group scheme over $\mathcal{O}_K$, we define $\deg(G) = \ell(\omega_G)/e_K$, where, ω_G is the $\mathcal{O}_K$-module of global invariant differentials on G, ℓ denotes the length of a module, and e_K is the ramification index of K. In fact, $\deg(G)$ equals the p-adic valuation of a generator δ_G of $Fitt_0(\omega_G)$, the zeroth Fitting ideal of ω_G.

We record some lemmas which we will use later.

Lemma 1.2.2 **([8, lemme 4]).** — *Assume that* $0 \to G' \to G \to G'' \to 0$ *is an exact sequence of finite flat group schemes over* $\mathcal{O}_K$. *We have* $\deg(G) = \deg(G') + \deg(G'')$.

Lemma 1.2.3 **([8]).** — *Let* $\lambda : A \to B$ *be an isogeny of p-power degree between abelian schemes over* $S = \mathrm{Spec}(\mathcal{O}_K)$. *Let* G *be the kernel of* λ. *Let* $\omega_{A/S}$ *and* $\omega_{B/S}$ *denote the sheaves of invariant differentials of* A *and* B, *respectively. Then*

$$\deg(G) = \nu_p(\det(\lambda^* : \omega_{B/S} \to \omega_{A/S})).$$

In particular, if A *is an abelian scheme over* $\mathrm{Spec}(\mathcal{O}_K)$ *of dimension* g, *then* $\deg(A[p^n]) = ng$.

Remark 1.2.4. — The degree of an isogeny between abelian varieties is seldom equal to the degree of its kernel.

Proposition 1.2.5 **([8, corollaire 3]).** — *Let* G *and* G' *be two finite flat group schemes over* $S = \mathrm{Spec}(\mathcal{O}_K)$, *and* $\lambda : G \to G'$ *a morphism of group schemes which is generically an isomorphism. Then,* $\deg(G) \leqslant \deg(G')$ *with equality if and only if* λ *is an isomorphism.*

Proposition 1.2.6 **([15, lemme 2.3.4]).** — *If* G *is a truncated Barsotti-Tate group of level* 1 *defined over a finite extension of* $\mathbb{Q}_p$, *then* $\deg(G)$ *is an integer.*

The degree function can be used to parameterize points on the modular curve, and to cut out rigid analytic subdomains on it.

Definition 1.2.7. — Let $Q = (\underline{E}, H)$ be a point on Y^{an}. If E has good reduction, we define $\deg(Q) = \deg(H)$. Otherwise, we define $\deg(Q) = 0$ or 1, depending on whether Q has étale or multiplicative reduction. If I is a subinterval of $[0,1]$, we define $Y^{\mathrm{an}}I$ to be the admissible open subdomain of Y^{an} consisting of points Q such that $\deg(Q) \in I$. If a, b are rational numbers, then $Y^{\mathrm{an}}[a,b]$ is quasi-compact. It is easy to see that the locus of supersingular reduction is exactly $Y^{\mathrm{an}}(0,1)$. The ordinary locus has two connected components, the multiplicative locus $Y^{\mathrm{an}}[1,1]$, and the étale locus, $Y^{\mathrm{an}}[0,0]$. An overconvergent modular form of weight $k \in \mathbb{Z}$ is a section of ω^k on $Y^{\mathrm{an}}[1-\epsilon, 1]$ for some $\epsilon > 0$.

Remark 1.2.8. — In Buzzard's work [**4**, § 4], the modular curve is parameterized by a function v' instead of deg. Roughly speaking, the value of v' at a supersingular point Q is the p-adic valuation of an appropriate parameter of the supersingular annulus containing Q. There is a simple relationship between deg and v': we have $v'(\underline{E}, H) = 1 - \deg(\underline{E}, H)$.

Given the above lemma, we can now rephrase the classical theory of canonical subgroups (due to Katz and Lubin) in terms of degrees, as follows:

***Proposition 1.2.9* (Lubin-Katz).** — *Let* $Q = (\underline{E}, H) \in Y^{\mathrm{an}}$. *Define*

$$\mathrm{Sib}(Q) = \{Q' = (\underline{E}, H') \in Y^{\mathrm{an}} : Q' \neq Q\}.$$

- *If* $\deg(Q) > 1/(p+1)$, *then, for any* $Q' \in \mathrm{Sib}(Q)$, *we have* $\deg(Q') = (1 - \deg(Q))/p < 1/(p+1)$.
- *If* $\deg(Q) = 1/(p+1)$, *then, for any* $Q' \in \mathrm{Sib}(Q)$, *we have* $\deg(Q') = 1/(p+1)$.
- *If* $\deg(Q) < 1/(p+1)$, *then, there is a unique* $(E, H') = Q' \in \mathrm{Sib}(Q)$, *such that* $\deg(Q') > 1/(p+1)$; H' *is called the (first) canonical subgroup of* $\underline{E}$, *it varies analytically with respect to* Q, *and we have* $\deg(Q') = 1 - p\deg(Q)$. *For all other* $Q'' \in \mathrm{Sib}(Q)$, *we have* $\deg(Q'') = \deg(Q) < 1/(p+1)$.

We make a definition:

Definition 1.2.10. — If $\deg(\underline{E}, H) < \frac{1}{p^{m-1}(p+1)}$, then, for any $1 \leqslant n \leqslant m$, we can define a cyclic subgroup C_n of $E[p^n]$ of order p^n, called the n-th canonical subgroup of E, inductively as follows. By Proposition 1.2.9, E has a first canonical subgroup C_1, and $\deg(\underline{E}/C_1, \bar{H}) = 1 - \deg(\underline{E}, C_1) = p\deg(\underline{E}, H) < \frac{1}{p^{m-2}(p+1)}$. Hence, by induction, we can construct C'_n, the n-th canonical subgroup of E/C_1, for all $1 \leqslant n \leqslant m-1$. For $2 \leqslant n \leqslant m$, we define $C_n = \mathrm{pr}^{-1}(C'_{n-1})$, where $\mathrm{pr} : E \to E/C_1$ is the projection.

The first step of the analytic continuation—the first take. — This step is due to Buzzard [**4**]. Using an iteration of the U_p operator, Buzzard extends f from its initial domain of definition to progressively larger domains, eventually extending f to $Y^{\mathrm{an}}(0, 1]$.

***Proposition 1.2.11* (Buzzard).** — *Let* f *be an overconvergent modular form* f *satisfying* $U_p(f) = a_p f$ *with* $a_p \neq 0$. *Then* f *extends analytically to* $Y^{\mathrm{an}}(0, 1]$.

We first recall the definition of the U_p operator. Let $\mathcal{V}$ and $\mathcal{W}$ be admissible opens of Y^{an} such that $\pi_1^{-1}(\mathcal{V}) \subset \pi_2^{-1}(\mathcal{W})$ inside $Y^{\mathrm{an},0}$. We define an operator

$$U_p = U_{\mathcal{W}}^{\mathcal{V}} : \omega^k(\mathcal{W}) \to \omega^k(\mathcal{V}),$$

via the formula

$$U_p f = \frac{1}{p}\pi_{1,*}(\mathrm{res}(\mathrm{pr}^* \pi_2^*(f))), \tag{1.2.1}$$

where res is restriction from $\pi_2^{-1}(\mathcal{W})$ to $\pi_1^{-1}(\mathcal{V})$, $\pi_{1,*}$ is the trace map associated with the finite flat map π_1, and $\mathrm{pr}^* : \pi_2^*\omega^k \to \pi_1^*\omega^k$ is a morphism of sheaves on Y^{an} which at $(\underline{E}, H, D)$ is induced by $\mathrm{pr}^* : \Omega_{E/D} \to \Omega_E$ coming from the natural projection $\mathrm{pr} : E \to E/D$.

One can also define a set-theoretic U_p correspondence as the map which sends a subset $S \subset Y^{\mathrm{an}}$ to another subset $U_p(S) = \pi_2(\pi_1^{-1}(S))$. The condition $\pi_1^{-1}(\mathcal{V}) \subset \pi_2^{-1}(\mathcal{W})$ is equivalent to $U_p(\mathcal{V}) \subset \mathcal{W}$.

The principle underlying Buzzard's method is the following. Let $\mathcal{W}$ be an admissible open such that $U_p(\mathcal{W}) \subset \mathcal{W}$. Suppose f is defined over $\mathcal{W}$ and $U_p(f) = a_p f$ with $a_p \neq 0$. Suppose further that $\mathcal{V} \supset \mathcal{W}$ is an admissible open subset of Y^{an} such that $U_p(\mathcal{V}) \subset \mathcal{W}$. Then, f extends from $\mathcal{W}$ to $\mathcal{V}$, and the extended section (which we