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ON OVERCONVERGENT HILBERT MODULAR CUSP FORMS

by

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Abstract. — We p-adically interpolate modular invertible sheaves over a strict neighborhood of the ordinary locus of an Hilbert modular variety. We then prove the existence of finite slope families of cuspidal eigenforms.

Résumé **(À propos des formes modulaires surconvergentes cuspidales de Hilbert)**
Nous interpolons p-adiquement les faisceaux inversibles automorphes sur des voisinages stricts du lieu ordinaire d'une variété modulaire de HIlbert. Nous prouvons ensuite l'existence de familles de pente finie de formes propres et cuspidales.

1. Introduction

Let F be a totally real number field. The theme of this paper is the construction of eigenvarieties for Hilbert modular eigenforms defined over F. Several constructions already appeared in the literature:

- A construction by Buzzard in [**9**], where he interpolates automorphic functions on a quaternion algebra over F, ramified at infinity.
- A construction by Kisin-Lai in [**19**] of a 1-dimensional parallel weight eigenvariety. Their method is an extension of Coleman's original construction for $F = \mathbb{Q}$. It is based on twists by a lift of the Hasse invariant.
- A construction by Urban in [**25**], who interpolates the traces of the Hecke operators acting on the Betti cohomology of certain distribution sheaves defined over the Hilbert variety (Urban actually works in a much more general setting).
- A yet conjectural construction by Emerton in [**14**] who applies a Jaquet functor to the completed cohomology of the Hilbert variety.
- The work of Kassaei in [**18**] for unitary/quaternionic Shimura curves as a consequence of which a 1 dimensional eigenvariety for Hilbert modular forms was constructed (for weights of the form $(k, 2, 2, \ldots, 2)$, where k-varies.)
- A recent construction of Brasca in [**8**] who also constructs a 1-dimensional eigenvariety for Hilbert modular forms using Shimura curves.

In the present paper we work with the Hilbert variety and p-adically interpolate modular invertible sheaves over a strict neighborhood of the ordinary locus. We then prove the existence of finite slope families of cuspidal eigenforms. Thus our construction is geometric, in the sense that we work over (open subsets of) the Hilbert variety and interpolate sections of invertible sheaves. In the ordinary case, our construction boils down to the construction of Hida families using Katz's p-adic modular forms. In the parallel weight case, our construction is equivalent to Kisin-Lai's (and the cuspidal hypothesis is unnecessary).

This article is a natural continuation of both [**3**] and [**2**] and we would like to discuss what is new here. Let us first point out that the notion of "Hilbert modular form" is slightly ambiguous in the literature if $F \neq \mathbb{Q}$. More precisely, given the totally real number field F as above, there are two relevant algebraic groups associated to it: $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ and $G^* := G \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m} \mathbb{G}_m$, where the morphism $G \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the determinant morphism and the morphism $\mathbb{G}_m \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the natural (diagonal) one. The Hilbert modular varieties considered in [**3**] and in section § 3 of this article are models for the Shimura varieties associated to the group G^*, therefore the classical, overconvergent and p-adic families of Hilbert modular forms considered there are all associated to the group G^*. Let us also point out that there are classical Hecke operators acting on those modular forms, but their definition depends on non-canonical choices and therefore these operators do not commute (although the U_p-operator is canonical.)

On the other hand there is a classical theory of automorphic forms for the group G, with a natural theory of eigenforms and to these eigenforms one can attach Galois representations. Moreover, recently D. Barrera in [**5**] showed that overconvergent modular symbols for the group G can be used in order to produce p-adic L-functions attached to classical automorphic eigenforms for this group. Therefore in view of various arithmetic applications it would be desirable to define overconvergent and p-adic families of modular forms for G. The main obstacle in doing this directly is that the moduli problem associated to G is not representable. If $F = \mathbb{Q}$, the ambiguity disappears as $G = G^\star$. As a result, we suppose that $F \neq \mathbb{Q}$ whenever we speak about the group G in the paper.

Let us now describe what is actually accomplished in this article.

a) In [**3**] we assumed that the prime p was unramified in F; in the present article we remove this assumption.

b) We start by constructing overconvergent and p-adic families of modular forms attached to the group G^*. As was mentioned before the construction is geometric but we do not follow the line of arguments started in [**3**]. Instead and for the sake of uniformity, we use the main ideas which appeared in [**2**].

c) We prove that the specialization map from finite slope p-adic families of cuspforms to finite slope overconvergent cuspforms of a given weight (still for the group G^*) is surjective, i.e., every overconvergent cuspform of finite slope can be deformed to a p-adic family. This was not proved in [**3**] even in the case when p was unramified in F.

d) We construct overconvergent and p-adic families of modular forms for the group G by descent, using the overconvergent and p-adic families of modular forms for G^* constructed at b).

e) We construct the cuspidal eigenvariety for the group G and study the associated Galois representations.

We shall now explain the main ideas of these constructions. For simplicity of the exposition, throughout the introduction we will only work with the open Hilbert modular varieties, but the reader should be aware that the shaves we construct extend to toroidal compactifications as will be explained in the main body of the article.

a) Let $N \geq 4$ be an integer prime to p, let $\mathfrak{c}$ denote a fractional ideal of F and $\mathfrak{c}^+$ its cone of positive elements. We fix K a finite extension of $\mathbb{Q}_p$ which splits F. We denote by $M(\mu_N, \mathfrak{c})$ the Hilbert modular scheme over $\mathrm{Spec}(\mathcal{O}_K)$ (for the group G^*) classifying abelian schemes of relative dimension $g := [F : \mathbb{Q}]$, level structure μ_N, $\mathcal{O}_F$-multiplication and polarization associated to $(\mathfrak{c}, \mathfrak{c}^+)$ (see section §3.1 for more details). Let $A \longrightarrow M(\mu_N, \mathfrak{c})$ be the universal abelian scheme with identity section $e \colon M(\mu_N, \mathfrak{c}) \longrightarrow A$ and let $\omega_A := e^*\big(\Omega^1_{A/M(\mu_N,\mathfrak{c})}\big)$ denote the co-normal sheaf of A. The sheaf ω_A is a locally free $\mathcal{O}_{M(\mu_N,\mathfrak{c})}$-module of rank g, with a natural action of $\mathcal{O}_F$ but if p ramifies in F it is not locally free as $\mathcal{O}_{M(\mu_N,\mathfrak{c})} \otimes \mathcal{O}_F$-module. There is a largest open subscheme $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$ of $M(\mu_N, \mathfrak{c})$ such that the restriction of ω_A to it is an invertible $\mathcal{O}_{M^{\mathrm{R}}(\mu_N,\mathfrak{c})} \otimes \mathcal{O}_F$-module. Therefore the classical Hilbert modular forms are in this case defined as sections of the relevant sheaf over the open $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$ (see section § 2.)

On the other hand the construction of the overconvergent Hilbert modular forms starts by fixing an integer $n > 0$, a multi-index $\underline{v} := (v_i)$ satisfying $0 < v_i < 1/p^n$ and a strict neighborhood $\mathcal{M}(\mu_n, \mathfrak{c})(\underline{v})$ of width $\underline{v}$ of the ordinary locus in the rigid analytic variety $\mathcal{M}(\mu_N, \mathfrak{c})$ associated to the generic fiber of $M(\mu_N, \mathfrak{c})$. Let us denote by $\mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v})$ the natural formal model of $\mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$ described in section §3.2.1

It has the property that the universal abelian scheme $A_{\underline{v}}$ on it has a canonical subgroup H_n of order p^{ng}. We denote by $\mathcal{M}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$ the finite étale covering of $\mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$ on which H_n is trivialized and by $\mathfrak{M}\big(\Gamma_1(p^n), \mu_N, \mathfrak{c}\big)(\underline{v})$ the normalization of $\mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v})$ in it. It turns out that over $\mathcal{M}(\Gamma_1(p^n), \mu_N, \mathfrak{c})(\underline{v})$, the canonical subgroup H_n is isomorphic to the constant group scheme $\mathcal{O}_F/p^n\mathcal{O}_F$. This isomorphism is compatible with the natural $\mathcal{O}_F$-actions on the two group schemes.

Therefore the sub-sheaf $\mathcal{F}$ of $\omega_A|_{\mathfrak{M}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})}$ defined in Proposition 5.1 is a locally free $\mathcal{O}_{\mathfrak{M}(\Gamma_1(p^n),\mu_N,\mathfrak{c})(\underline{v})} \otimes \mathcal{O}_F$-module of rank one while the restriction of ω_A is not (if p is ramified in F).

It follows that the overconvergent modular sheaves are better behaved than the classical ones and the overconvergent modular forms are defined as sections of the relevant modular sheaves over opens of the form $\mathfrak{M}(\mu_N, \mathfrak{c})(\underline{v})$ wether p is ramified in F or not.

b) In **[3]** we defined modular sheaves by using certain universal torsors while in this article we content ourselves to only define their realization on the universal formal

Hilbert modular schemes. Instead of torsors we use overconvergent Igusa towers as in [**2**]. The present constructions are more restrictive but they suffice for the applications we have in mind.

c) The surjectiveness of the specialization map is proved as in [**2**]: we study the descent of our modular sheaves from a smooth toroidal compactification $\overline{\mathcal{M}}(\mu_N, \mathfrak{c})(\underline{v})$ of $\mathcal{M}(\mu_N, \mathfrak{c})(\underline{v})$ to the minimal compactification $\overline{\mathcal{M}}^*(\mu_N, \mathfrak{c})(\underline{v})$.

d) We shall first recall here the descent from the classical modular forms for the group G^* to modular forms for the group G; the latter are known to be identified, with their Hecke operators, to the classical automorphic forms for $G(\mathbb{A}_F)$, see for instance [**21**], section § 2.

Let us recall the notations at a) above: $N \geq 4$ is an integer prime to p, $\mathfrak{c}$ denotes a fractional ideal of F and $\mathfrak{c}^+$ its cone of positive elements. We denote by $M(\mu_N, \mathfrak{c})$ the Hilbert modular scheme over $\mathrm{Spec}(\mathcal{O}_K)$ (for the group G^*) classifying abelian schemes of relative dimension $g := [F : \mathbb{Q}]$, level structure μ_N, $\mathcal{O}_F$-multiplication and polarization associated to $(\mathfrak{c}, \mathfrak{c}^+)$ (see section §3.1 for more details). Let $A \longrightarrow M(\mu_N, \mathfrak{c})$ be the versal abelian scheme and let ω_A be the co-normal sheaf to the identity of A. We denoted by $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$ the largest open subscheme of $M(\mu_n, \mathfrak{c})$ such that the restriction of ω_A to it is an invertible $\mathcal{O}_{M^{\mathrm{R}}(\mu_N,\mathfrak{c})} \otimes \mathcal{O}_F$-module.

We denote by $\mathbb{T}$ the algebraic group $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_m$ over $\mathcal{O}_K$. This group is identified with the diagonal subgroup inside the derived group $G^{\star,der} = G^{der}$. Let $\mathbb{T}^\star$ be the diagonal subgroup in $G^\star$ and let $\mathbb{G}_m$ be the multiplicative group identified with the center of $G^\star$. The surjective map $\mathbb{T}\times\mathbb{G}_m \to \mathbb{T}^\star$ has kernel the group μ_2. Let I be the set of embeddings of F into K (recall that K was a finite extension of $\mathbb{Q}_p$ which splits F). Then $\mathbb{Z}[I]$ is the character group, over K, of the group $\mathbb{T}$. If we identify $\mathbb{Z}$ with the character group of $\mathbb{G}_m$, then the character group of $\mathbb{T}^\star$ is the subgroup of $\mathbb{Z}[I] \times \mathbb{Z}$ of elements $(\sum_{\sigma\in I} k_\sigma.\sigma, w)$ such that $\sum k_\sigma = w \mod 2$. According to [**20**], III.2, V.6., we can attach to each $(\kappa, w) \in \mathbb{Z}[I]\times\mathbb{Z}$ an invertible sheaf $\Omega_K^{(\kappa,w)}$ over $M(\mu_N, \mathfrak{c})_K$. Here we are using that the center of $G^\star$ is $\mathbb{G}_m$ and that $\mathbb{G}_m(\mathbb{Z})$ is a finite group. Actually, this sheaf does not depend on w (which must be considered like a Tate twist), so we suppress it from the notation as this is customary. This construction of the sheaf can be made explicit and extended integrally as follows. Let $\kappa := \sum_{\sigma\in I} k_\sigma \cdot \sigma \in \mathbb{Z}[I]$ and let us set $\Omega^\kappa := \bigoplus_{\sigma\in I} \omega_{A,\sigma}^{k_\sigma}$. The notation $\omega_{A,\sigma}$ means the sheaf $\omega_A \otimes_{(\mathcal{O}_K\otimes\mathcal{O}_F, 1\otimes\sigma)} \mathcal{O}_K$ over $M^{\mathrm{R}}(\mu_N, \mathfrak{c})$. We refer to $\mathbb{Z}[I]$ as the set of *classical weights* for $G^\star$.

The module $\mathrm{M}(\mu_N, \mathfrak{c}, \kappa) := \mathrm{H}^0\big(M^{\mathrm{R}}(\mu_N, \mathfrak{c})_K, \Omega^\kappa\big)$ is the K-vector space of tame level N, $\mathfrak{c}$-polarized, weight κ Hilbert modular forms (for the group G^*).

Let $\mathbb{T}^G$ be the diagonal subgroup in G. We have a map $p_1 : \mathbb{T} \to \mathbb{T}^G$ given by the inclusion $G^\star \to G$. We can also identify $\mathbb{T}$ with the center of G. This provides a second map $p_2 : \mathbb{T} \to \mathbb{T}^G$. The map $p_1 \times p_2 : \mathbb{T} \times \mathbb{T} \to \mathbb{T}_G$ is surjective, with kernel $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mu_2$. As a result, the character group over K of $\mathbb{T}^G$ is the subgroup of $\mathbb{Z}[I]\times\mathbb{Z}[I]$ of pairs $(\sum_\sigma k_\sigma.\sigma, \sum_\sigma w_\sigma.\sigma)$ subject to the condition that $k_\sigma = w_\sigma \mod 2$. The map from characters of $\mathbb{T}^G$ to characters of $\mathbb{T}^\star$ is the one that sends $(\sum_\sigma k_\sigma.\sigma, \sum_\sigma w_\sigma.\sigma)$