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p-ADIC COHOMOLOGY AND CLASSICALITY OF OVERCONVERGENT HILBERT MODULAR FORMS

by

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Abstract. — Let F be a totally real field in which a prime number p is unramified. We prove that, if a cuspidal overconvergent Hilbert modular form has small slopes under the U_p-operators, then it is classical. Our method follows the original cohomological approach of R. Coleman. The key ingredient of the proof is giving an explicit description of the Goren-Oort stratification of the special fiber of the Hilbert modular variety. As a byproduct of the proof, we show that, at least when p is inert, the rigid cohomology of the ordinary locus is equal to the space of classical forms in the Grothendieck group of finite-dimensional modules of the Hecke algebras.

Résumé **(Cohomologie p-adique et classicité de formes modulaires surconvergentes de Hilbert)**
Soit F un corps totalement réel dans lequel un nombre premier p est non ramifié. Nous prouvons que toute forme cuspidale surconvergente de Hilbert de petite pente pour les opérateurs U_p est classique. Notre méthode suit l'approche cohomologique originelle de R. Coleman. L'ingrédient-clé de la preuve est fourni par une description explicite de la stratification de Goren-Oort de la fibre spéciale de la variété de Hilbert. Comme corollaire de la démonstration, nous montrons que lorsque p est inerte, la cohomologie rigide du lieu ordinaire est égale à l'espace des formes classiques dans le groupe de Grothendieck des modules de dimension finie sur l'algèbre de Hecke.

1. Introduction

The classicality results for p-adic overconvergent modular forms started with the pioneering work of R. Coleman [**14**], in which he proved that an overconvergent modular form of weight k and slope $< k-1$ is classical. Coleman proved his theorem using p-adic cohomology and an ingenious dimension counting argument. Later, P. Kassaei [**28**] reproved Coleman's theorem based on an analytic continuation result by K. Buzzard [**10**]. In the Hilbert case, S. Sasaki [**47**] proved the classicality of small slope overconvergent Hilbert modular forms when the prime p is totally split in the concerning totally real field. With a less optimal slope condition, such a classicality result

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for overconvergent Hilbert modular forms was proved by the first named author [**51**] in the quadratic inert case, and by V. Pilloni and B. Stroh in the general unramified case [**26**]. The methods of [**47, 51, 26**] followed that of Kassaei, and used the analytic continuation of overconvergent Hilbert modular forms.

In this paper, we will follow Coleman's original cohomological approach to prove the classicality of cuspidal overconvergent Hilbert modular forms. Let us describe our main results in details. We fix a prime number p. Let F be a totally real field of degree $g = [F:\mathbb{Q}] \geq 2$ in which p is unramified, and denote by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the primes of F above p. Let Σ_∞ be the set of archimedean places of F. We fix an isomorphism $\iota_p : \mathbb{C} \cong \overline{\mathbb{Q}}_p$. For each $\mathfrak{p}_i$, we denote by $\Sigma_{\infty/\mathfrak{p}_i}$ the subset of archimedean places $\tau \in \Sigma_\infty$ such that $\iota_p \circ \tau$ induce the prime $\mathfrak{p}_i$. We fix an ideal $\mathfrak{N}$ of $\mathcal{O}_F$ coprime to p. We consider the following level structures:

$$K_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \;\Big|\; a \equiv 1, c \equiv 0 \bmod \mathfrak{N} \right\};$$

$$K_1(\mathfrak{N})^p\mathrm{Iw}_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(\mathfrak{N}) \;\Big|\; c \equiv 0 \bmod p \right\}.$$

Consider a multi-weight $(\underline{k}, w) \in \mathbb{N}^{\Sigma_\infty} \times \mathbb{N}$ such that $w \geq k_\tau \geq 2$ and $k_\tau \equiv w \pmod 2$ for all τ (such a multi-weight will be called cohomological). The convention on weights in this paper is adapted for arithmetic applications: each archimedean component of the automorphic representation associated to a cuspidal Hilbert eigenform of multi-weight $(\underline{k}, w)$ has central character $t \mapsto t^{w-2}$. This agrees with [**46**]. Our first main theorem is the following:

Theorem 1 **(Theorem 6.9).** — *Let f be a cuspidal overconvergent Hilbert modular form of multiweight $(\underline{k}, w)$ and level $K_1(\mathfrak{N})$, which is an eigenform for all Hecke operators. Let $\lambda_{\mathfrak{p}_i}$ denote the eigenvalue of f for the operator $U_{\mathfrak{p}_i}$ for $1 \leq i \leq r$. If the p-adic valuation of each $\lambda_{\mathfrak{p}_i}$ satisfies*

$$\mathrm{val}_p(\lambda_{\mathfrak{p}_i}) < \sum_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} \frac{w - k_\tau}{2} + \min_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} \{k_\tau - 1\}, \tag{1.0.1}$$

then f is a classical (cuspidal) Hilbert eigenform of level $K_1(\mathfrak{N})^p\mathrm{Iw}_p$.

Here, we normalize the p-adic valuation val_p so that $\mathrm{val}_p(p) = 1$. The term $\sum_\tau \frac{w-k_\tau}{2}$ is a normalizing factor that appears in the definition of cuspidal overconvergent Hilbert modular forms. Any cuspidal overconvergent Hilbert eigenform has $U_{\mathfrak{p}_i}$-slope greater than or equal to this quantity. Up to this normalizing factor, Theorem 1 was proved in [**26**] (and also in [**51**] for the quadratic case) with slope bound $\mathrm{val}_p(\lambda_{\mathfrak{p}_i}) < \sum_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} \frac{w-k_\tau}{2} + \min_{\tau \in \Sigma_{\infty/\mathfrak{p}_i}} (k_\tau - [F_{\mathfrak{p}_i} : \mathbb{Q}_p])$. The slope bound (1.0.1), believed to be optimal, was conjectured by Breuil in an unpublished note [**8**], which significantly inspires this work. In Theorem 6.9, we also give some classicality results using theta operators if the slope bound (1.0.1) is not satisfied, as conjectured by Breuil in *loc.*

cit. Finally, Christian Johansson [**27**] also obtained independently in his thesis similar results for overconvergent automorphic forms for rank two unitary group, but with a less optimal slope bound.

We now explain the proof of our theorem. As in [**14**], the first step is to relate the cuspidal overconvergent Hilbert modular forms to a certain p-adic cohomology group of the Hilbert modular variety.

We take the level structure $K = K^p K_p$ to be hyperspecial at places above p. Let $K^p\mathrm{Iw}_p$ denote the corresponding level structure with the same tame level K^p and with Iwahori group at all places above p. Let $\mathbf{X}$ be the integral model of the Hilbert modular variety of level K defined over the ring of integers of a finite extension L over $\mathbb{Q}_p$. We choose a toroidal compactification $\mathbf{X}^{\mathrm{tor}}$ of $\mathbf{X}$. Let X^{tor} and X denote respectively the special fibers of $\mathbf{X}^{\mathrm{tor}}$ and $\mathbf{X}$ over $\overline{\mathbb{F}}_p$, and D be the boundary $X^{\mathrm{tor}} - X$. Let $X^{\mathrm{tor,ord}}$ be the ordinary locus of X^{tor}. Let $\mathscr{F}^{(\underline{k},w)}$ denote the corresponding overconvergent log-F-isocrystal sheaf of multiweight $(\underline{k},w)$ on X^{tor}, and let $S^{\dagger}_{(\underline{k},w)}$ denote the space of cuspidal overconvergent Hilbert modular forms. We consider the rigid cohomology of $\mathscr{F}^{(\underline{k},w)}$ over the ordinary locus of X^{tor} with compact support at cusps, denoted by $H^{\star}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ (see Subsection 3.4 for its precise definition). Using the dual BGG-complex and a cohomological computation due to Coleman [**14**], we show in Theorem 3.5 that, the cohomology group above is computed by a complex consisting of cuspidal overconvergent Hilbert modular forms.

Let us explain more explicitly this result in the case when F is a real quadratic field and p is a prime inert in $F/\mathbb{Q}$. Then Theorem 3.5 says that the cohomology group $H^{\star}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ (together with its Hecke action) is computed by the complex

$$\mathscr{C}^{\bullet}\colon\ S^{\dagger}_{(2-k_1,2-k_2,w)} \xrightarrow{(\Theta_1,\Theta_2)} S^{\dagger}_{(k_1,2-k_2,w)} \oplus S^{\dagger}_{(2-k_1,k_2,w)} \xrightarrow{-\Theta_2\oplus\Theta_1} S^{\dagger}_{(k_1,k_2,w)},$$

where each Θ_i is essentially the (k_i-1)-times composition of the Hilbert analogues of the well-known θ-operator for the elliptic modular forms. We refer the reader to Subsection 2.15 and Remark 2.17 for the precise expression of Θ_i's, and to (3.3.2) for the definition of the complex $\mathscr{C}^{\bullet}$ in the general case. Here, the Hecke action on its terms $S^{\dagger}_{\star}$ coincides with the one given in [**33**] (see Remark 3.19 for details), and the complex $\mathscr{C}^{\bullet}$ is Hecke equivariant for this Hecke action.

An important fact for us is that, the slope condition (1.0.1) can be satisfied only for eigenforms in the last term $S^{\dagger}_{(k_1,k_2,w)}$. In other words, if an eigenform $f \in S^{\dagger}_{(k_1,k_2,w)}$ satisfies the slope condition, then it has nontrivial image in the cohomology group $H^{g}_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$. This result on U_p-action is explained in Corollary 3.24.

Moreover, the cohomological approach allows us to prove the following strengthened version of Theorem 1: if a cuspidal overconvergent Hilbert modular form f of multiweight $(\underline{k},w)$ and level K does not lie in the image of all Θ-maps, then f is a classical (cuspidal) Hilbert modular form.

The second step of the proof of Theorem 1 is to compute $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ using the Goren-Oort stratification of X. A key ingredient here is the explicit description of these Goren-Oort strata of X given in [52]. In the quadratic inert case considered above, the main results of [52] can be described as follows. Let X_1 and X_2 be respectively the vanishing loci of the two partial Hasse invariants on X^{tor}. Then according to [20], $X_1 \cup X_2$ a normal crossing divisor of X^{tor}, and it is complement of the ordinary locus $X^{\mathrm{tor,ord}} \subseteq X^{\mathrm{tor}}$. Put $X_{12} = X_1 \cap X_2$. It was previously known that each of X_1 and X_2 is a certain collection of $\mathbb{P}^1$'s. The main result of [52] says that each of X_1 and X_2 is isomorphic a $\mathbb{P}^1$-bundle over $\mathbf{Sh}_K(B^\times_{\infty_1,\infty_2})_{\mathbb{F}_p}$, the special fiber of the discrete Shimura variety of level K associated to the quaternion algebra B_{∞_1,∞_2} over F which ramifies exactly at both archimedean places. Their intersection X_{12} may be identified with the Shimura variety $\mathbf{Sh}_{K^p\mathrm{Iw}_p}(B^\times_{\infty_1,\infty_2})_{\mathbb{F}_p}$ for the same group but with Iwahori level structure at p. Moreover, these isomorphisms are compatible with the tame Hecke actions.

In the general case, for each subset $\mathtt{T} \subset \Sigma_\infty$, we consider the closed Goren-Oort stratum $X_{\mathtt{T}}$ defined as the vanishing locus of the partial Hasse invariants corresponding to $\mathtt{T}$. This is a proper and smooth closed subvariety of X of codimension $\#\mathtt{T}$ by [20]. The main result of [52] shows that $X_{\mathtt{T}}$ is a certain $(\mathbb{P}^1)^N$-bundle over the special fiber of another quaternionic Shimura variety. In fact, this result is more naturally stated for the Shimura variety associated to the group $\mathrm{GL}_2(F) \times_{F^\times} E^\times$ with E a quadratic CM extension of F. We refer the reader to Section 5 for a more detailed discussion. Using this result and the Jacquet-Langlands correspondence, one can compute the cohomology of each closed Goren-Oort stratum. General formalism of rigid cohomology then produces a spectral sequence which relates the desired cohomology group $H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})$ to the cohomology of the closed Goren-Oort strata. In the general case, we prove the following

***Theorem 2* (Theorems 3.5 and 6.1).** — *We have the following equalities in the Grothendieck group of finite-dimensional modules of the* tame *Hecke algebra $\mathscr{H}(K^p, L)$:*

$$\sum_{J \subseteq \Sigma_\infty} (-1)^{\#J}\big[(S^\dagger_{(s_J\cdot\underline{k},w)})^{\mathrm{slope}\leq T}\big] = \big[H^\star_{\mathrm{rig}}(X^{\mathrm{tor,ord}}, \mathsf{D}; \mathscr{F}^{(\underline{k},w)})\big] = (-1)^g\big[S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)\big],$$

for T sufficiently large, where

- *$s_J \cdot \underline{k} \in \mathbb{Z}^{\Sigma_\infty}$ is the multi-weight whose τ-component is k_τ for $\tau \in J$, and is $2 - k_\tau$ for $\tau \notin J$;*
- *the superscript slope $\leq T$ means to take the* finite dimensional *subspace where the slope of the product of the $U_\mathfrak{p}$-operators is less than or equal to $T \in \mathbb{R}$; and*
- *$S_{(\underline{k},w)}(K^p\mathrm{Iw}_p)$ is the space of classical cuspidal Hilbert modular forms of level $K^p\mathrm{Iw}_p$.*

At this point, there are two ways to proceed to get Theorem 1. The first approach is unconditional. We first use Theorem 2 to prove the classicality result when the slope is much smaller the weight (Proposition 6.3). Then we improve the slope bound by