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SUBANALYTIC SHEAVES AND SOBOLEV SPACES

Construction of sheaves on the subanalytic site

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CONSTRUCTION OF SHEAVES ON THE SUBANALYTIC SITE

by

Stéphane Guillermou & Pierre Schapira

Abstract. — On a real analytic manifold M, we construct the linear subanalytic Grothendieck topology M_{sal} together with the natural morphism of sites ρ from M_{sa} to M_{sal}, where M_{sa} is the usual subanalytic site. Our first result is that the derived direct image functor by ρ admits a right adjoint, allowing us to associate functorially a sheaf (in the derived sense) on M_{sa} to a presheaf on M_{sa} satisfying suitable properties, this sheaf having the same sections that the presheaf on any open set with Lipschitz boundary. We apply this construction to various presheaves on real manifolds, such as the presheaves of functions with temperate growth of a given order at the boundary or with Gevrey growth at the boundary. (In a separated paper, Gilles Lebeau will use these techniques to construct the Sobolev sheaves.) On a complex manifold endowed with the subanalytic topology, the Dolbeault complexes associated with these new sheaves allow us to obtain various sheaves of holomorphic functions with growth. As an application, we can endow functorially regular holonomic $\mathscr{D}$-modules with a filtration, in the derived sense.

Résumé **(Construction de faisceaux sur le site sous-analytique).** — Sur une variété analytique réelle M nous construisons la topologie de Grothendieck linéaire M_{sal} et le morphisme naturel de sites ρ de M_{sa} vers M_{sal}, où M_{sa} est le site sous-analytique usuel. Notre premier résultat est que le foncteur dérivé de l'image directe par ρ admet un adjoint à droite, ce qui nous permet d'associer fonctoriellement un faisceau (au sens dérivé) sur M_{sa} à un préfaisceau sur M_{sa} satisfaisant certaines propriétés, ce faisceau ayant les mêmes sections que le préfaisceau sur tout ouvert à bord Lipschitz. Nous appliquons cette construction à divers préfaisceaux sur des variétés réelles, tels que le préfaisceau des fonctions à croissance tempérée d'un ordre donné le long du bord ou à croissance Gevrey le long du bord. (Dans un article séparé, Gilles Lebeau utilisera ces techniques pour construire les faisceaux de Sobolev.) Sur une variété complexe munie de la topologie sous-analytique, les complexes de Dolbeault associés à ces nouveaux faisceaux nous permettent d'obtenir divers faisceaux de fonctions holomorphes à croissance. Comme application, nous pouvons munir fonctoriellement les $\mathscr{D}$-modules holonomes réguliers d'une filtration, au sens dérivé.

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Introduction

Let M be a real analytic manifold. The Grothendieck subanalytic topology on M, denoted M_{sa}, and the morphism of sites $\rho_{\mathrm{sa}}\colon M \to M_{\mathrm{sa}}$, were introduced in [**13**]. Recall that the objects of the site M_{sa} are the relatively compact subanalytic open subsets of M and the coverings are, roughly speaking, the finite coverings. In loc. cit. the authors use this topology to construct new sheaves which would have no meaning on the usual topology, such as the sheaf $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ of $\mathscr{C}^{\infty}$-functions with temperate growth and the sheaf $\mathscr{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ of temperate distributions. On a complex manifold X, using the Dolbeault complexes, they constructed the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ (in the derived sense) of holomorphic functions with temperate growth. The last sheaf is implicitly used in the solution of the Riemann-Hilbert problem by Kashiwara [**8, 9**] and is also extremely important in the study of irregular holonomic $\mathscr{D}$-modules (see [**14**, § 7]).

In this paper, we shall modify the preceding construction in order to obtain sheaves of $\mathscr{C}^{\infty}$-functions with a given growth at the boundary. For example, functions whose growth at the boundary is bounded by a given power of the distance (temperate growth of order $s \geq 0$), or by an exponential of a given power of the distance (Gevrey growth of order $s > 1$), as well as their holomorphic counterparts. For that purpose, we have to refine the subanalytic topology and we introduce what we call the linear subanalytic topology, denoted M_{sal}.

Let us describe the contents of this paper with some details.

In Chapter 1 we construct the linear subanalytic topology on M. Denoting by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the category of open relatively compact subanalytic subsets of M, the presite underlying the site M_{sal} is the same as for M_{sa}, namely $\mathrm{Op}_{M_{\mathrm{sa}}}$, but the coverings are the linear coverings. Roughly speaking, a finite family $\{U_i\}_{i\in I}$ is a linear covering of their union U if there is a constant C such that the distance of any $x \in M$ to $M \setminus U$ is bounded by C-times the maximum of the distance of x to $M \setminus U_i$ $(i \in I)$. (See Definition 1.1.) In this chapter, we also prove some technical results on linear coverings that we shall need in the course of the paper.

Chapter 2. Let $\mathbf{k}$ be a field. One easily shows that a presheaf F of $\mathbf{k}$-modules on M_{sal} is a sheaf as soon as, for any open sets U_1 and U_2 such that $\{U_1, U_2\}$ is a linear covering of $U_1 \cup U_2$, the Mayer-Vietoris sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \tag{0.1}$$

is exact. Moreover, if for any such a covering, the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0 \tag{0.2}$$

is exact, then the sheaf F is Γ-acyclic, that is, $\mathrm{R}\Gamma(U;F)$ is concentrated in degree 0 for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.

There is a natural morphism of sites $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ and we shall prove the two results below (see Theorems 2.30 and 2.49):

(1) the functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a right adjoint $\rho^!_{\mathrm{sal}}$,

(2) if U has a Lipschitz boundary, then the object $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U$ is concentrated in degree 0.

Therefore, if a presheaf F on M_{sa} has the property that the Mayer-Vietoris sequences (0.2) are exact, it follows that $\mathrm{R}\Gamma(U;\rho^!_{\mathrm{sal}}F)$ is concentrated in degree 0 and is isomorphic to $F(U)$ for any U with Lipschitz boundary. In other words, to a presheaf on M_{sa} satisfying a natural condition, we are able to associate an object of the derived category of sheaves on M_{sa} which has the same sections as F on any Lipschitz open set. This construction is in particular used by Gilles Lebeau [**21**] who obtains for $s \leq 0$ the "Sobolev sheaves $\mathscr{H}^s_{M_{\mathrm{sa}}}$," objects of $\mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ with the property that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has a Lipschitz boundary, then $\mathrm{R}\Gamma(U;\mathscr{H}^s_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$.

The fact that Sobolev sheaves are objects of derived categories and are not concentrated in degree 0 shows that when dealing with spaces of functions or distributions defined on open subsets which are not regular (more precisely, which have not a Lipschitz boundary), it is natural to replace the notion of a space by that of a complex of spaces.

In Chapter 3, we briefly study the natural operations on the linear subanalytic sites. The main difficulty is that a morphism $f\colon M \to N$ of real analytic manifolds does not induce a morphism of the linear subanalytic sites. This forces us to treat separately the direct or inverse images of sheaves for closed embeddings and for submersive maps.

In Chapter 4 we construct some sheaves on M_{sal}. We construct the sheaf $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ of $\mathscr{C}^\infty$-functions with growth of order $s \geq 0$ at the boundary and the sheaves $\mathscr{C}^{\infty,\mathrm{gev}(s)}_{M_{\mathrm{sal}}}$ and $\mathscr{C}^{\infty,\mathrm{gev}\{s\}}_{M_{\mathrm{sal}}}$ of $\mathscr{C}^\infty$-functions with Gevrey growth of type $s > 1$ at the boundary. By using a refined cut-off lemma (which follows from a refined partition of unity due to Hörmander [**6**]), we prove that these sheaves are Γ-acyclic. Applying the functor $\rho^!_{\mathrm{sal}}$, we get new sheaves (in the derived sense) on M_{sa} whose sections on open sets with Lipschitz boundaries are concentrated in degree 0. Then, on a complex manifold X, by considering the Dolbeault complexes of the sheaves of $\mathscr{C}^\infty$-functions considered above, we obtain new sheaves of holomorphic functions with various growth.

As already mentioned, Sobolev sheaves are treated in a separate paper by G. Lebeau in [**21**].

Finally, in Chapter 5, we apply these results to endow the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with a filtration (in the derived sense) that we call the L^∞-filtration.

Denote by $\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}$ the sheaf $\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M$ of differential operators on M_{sa}, endowed with its natural filtration and denote by $\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}}$ the sheaf $\mathscr{D}_{M_{\mathrm{sal}}} := \rho_{\mathrm{sal}*}\mathscr{D}_{M_{\mathrm{sa}}}$ endowed with its natural filtration. For $\mathscr{T} = M, M_{\mathrm{sa}}, M_{\mathrm{sal}}$, the category $\mathrm{Mod}(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ of filtered $\mathscr{D}$-modules on $\mathscr{T}$ is quasi-abelian in the sense of [**32**] and its derived category $\mathrm{D}^+(\mathrm{F}\mathscr{D}_{\mathscr{T}})$ is well-defined. We shall use here the recent results of [**31**] which give an easy description of these derived categories and we construct a right adjoint $\rho^!_{\mathrm{sal}}$ to the derived functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{M_{\mathrm{sal}}})$.

By considering the sheaves $\mathscr{C}^{\infty,s}_{M_{\mathrm{sal}}}$ $(s \geq 0)$ we obtain the filtered sheaf $\mathrm{F}_\infty\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}$. Then, on a complex manifold X, by considering the Dolbeault complex of this filtered sheaf, we obtain the filtration $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ on the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$.

Recall now the Riemann-Hilbert correspondence. Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module and let $G := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ be the perverse sheaf of its holomorphic solutions. Kashiwara's theorem of [9] may be formulated by saying that the natural morphism $\mathscr{M} \to \rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}om(G, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is an isomorphism. Replacing the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ with its filtered version $\mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$, we define the filtered Riemann-Hilbert functors $\mathrm{RHF}_{\infty,\mathrm{sa}}$ and RHF_∞ by the formulas

$$\begin{aligned} \mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_{X_{\mathrm{sa}}}), \\ \mathscr{M} &\mapsto \mathrm{FR}\mathscr{H}om(\mathrm{Sol}(\mathscr{M}), \mathrm{F}_\infty\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}), \\ \mathrm{RHF}_\infty = \rho_{\mathrm{sa}}^{-1}\mathrm{RHF}_{\infty,\mathrm{sa}}\colon \mathrm{D}^+_{\mathrm{holreg}}(\mathscr{D}_X) &\to \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \end{aligned}$$

and we prove that the composition

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathscr{D}_X) \xrightarrow{\mathrm{RHF}_\infty} \mathrm{D}^+(\mathrm{F}\mathscr{D}_X) \xrightarrow{\mathrm{for}} \mathrm{D}^+(\mathscr{D}_X)$$

is isomorphic to the identity functor. In other words, any regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$ can be *functorially* endowed with a filtration $\mathrm{F}_\infty\mathscr{M}$, in the derived sense.

We also briefly introduce an L^2-filtration better suited to apply Hörmander's theory (see [5]) and present some open problems.

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We have also been very much stimulated by the interest of Gilles Lebeau for sheafifying the classical Sobolev spaces and it is a pleasure to thank him here.

Finally Theorem 2.43 plays an essential role in the whole paper and we are extremely grateful to Adam Parusinski who has given a proof of this result.

1. Subanalytic topologies

1.1. Linear coverings

Notations and conventions. — We shall mainly follow the notations of [11, 13] and [15].

In this paper, unless otherwise specified, a manifold means a real analytic manifold. We shall freely use the theory of subanalytic sets, due to Gabrielov and Hironaka, after the pioneering work of Lojasiewicz. A short presentation of this theory may be found in [2].

For a subset A in a topological space X, $\overline{A}$ denotes its closure, $\mathrm{Int}\, A$ its interior and ∂A its boundary, $\partial A = \overline{A} \setminus \mathrm{Int}\, A$.