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Derived categories of filtered objects

Pierre Schapira & Jean-Pierre Schneiders

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
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DERIVED CATEGORIES OF FILTERED OBJECTS

by

Pierre Schapira & Jean-Pierre Schneiders

Abstract. — For an abelian category $\mathscr{C}$ and a filtrant preordered set Λ, we prove that the derived category of the quasi-abelian category of filtered objects in $\mathscr{C}$ indexed by Λ is equivalent to the derived category of the abelian category of functors from Λ to $\mathscr{C}$. We apply this result to the study of the category of filtered modules over a filtered ring in a tensor category.

***Résumé* (Catégories dérivées d'objets filtrés).** — Pour une catégorie abélienne $\mathscr{C}$ et un ensemble préordonné filtrant Λ, nous prouvons que la catégorie dérivée de la catégorie quasi-abélienne des objets filtrés de $\mathscr{C}$ indexés par Λ est équivalente à la catégorie dérivée de la catégorie abélienne des foncteurs de Λ dans $\mathscr{C}$. Nous appliquons ce résultat à l'étude de la catégorie des modules filtrés sur un anneau filtré d'une catégorie tensorielle.

1. Introduction

Filtered modules over filtered sheaves of rings appear naturally in mathematics, such as for example when studying $\mathscr{D}_X$-modules on a complex manifold X, $\mathscr{D}_X$ denoting the filtered ring of differential operators (see [**3**]). As it is well-known, the category of filtered modules over a filtered ring is not abelian, only exact in the sense of Quillen [**7**] or quasi-abelian in the sense of [**8**], but this is enough to consider the derived category (see [**1, 6**]). However, quasi-abelian categories are not easy to manipulate, and we shall show in this paper how to substitute a very natural abelian category to this quasi-abelian category, giving the same derived category.

More precisely, consider an abelian category $\mathscr{C}$ admitting small exact filtrant (equivalently, "directed") colimits and a filtrant preordered set Λ. In this paper, we regard a filtered object in $\mathscr{C}$ as a functor $M\colon \Lambda \to \mathscr{C}$ with the property that all $M(\lambda)$ are sub-objects of $\varinjlim M$. We prove that the derived category of the quasi-abelian

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category of filtered objects in $\mathscr{C}$ indexed by Λ is equivalent to the derived category of the abelian category of functors from Λ to $\mathscr{C}$. Note that a particular case of this result, in which $\Lambda = \mathbb{Z}$ and $\mathscr{C}$ is the category of abelian groups, was already obtained in [**8**, § 3.1].

Next, we assume that $\mathscr{C}$ is a tensor category and Λ is a preordered semigroup. In this case, we can define what is a filtered ring A indexed by Λ and a filtered A-module in $\mathscr{C}$ and we prove a similar result to the preceding one, namely that the derived category of the category of filtered A-modules is equivalent to the derived category of the abelian category of modules over the Λ-ring A.

Applications to the study of filtered $\mathscr{D}_X$-modules will be developed in the future. Indeed it is proved in [**2**] that, on a complex manifold X endowed with the subanalytic topology X_{sa}, the sheaf $\mathscr{O}_{X_{\mathrm{sa}}}$ (which is in fact an object of the derived category of sheaves, no more concentrated in degree zero) may be endowed with various filtrations and the results of this paper will be used when developing this point.

2. A review on quasi-abelian categories

In this section, we briefly review the main notions on quasi-abelian categories and their derived categories, after [**8**]. We refer to [**5**] for an exposition on abelian, triangulated and derived categories.

Let $\mathscr{C}$ be an additive category admitting kernels and cokernels. Recall that, for a morphism $f\colon X \to Y$ in $\mathscr{C}$, $\mathrm{Im}(f)$ is the kernel of $Y \to \mathrm{Coker}(f)$, and $\mathrm{Coim}(f)$ is the cokernel of $\mathrm{Ker}(f) \to X$. Then f decomposes as

$$X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y.$$

One says that f is *strict* if $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. Note that a monomorphism (resp. an epimorphism) $f\colon X \to Y$ is strict if and only if $X \to \mathrm{Im}(f)$ (resp. $\mathrm{Coim}(f) \to Y$) is an isomorphism. For any morphism $f\colon X \to Y$,

- $\mathrm{Ker}(f) \to X$ and $\mathrm{Im}(f) \to Y$ are strict monomorphisms,
- $X \to \mathrm{Coim}(f)$ and $Y \to \mathrm{Coker}(f)$ are strict epimorphisms.

Note also that a morphism f is strict if and only if it factors as $i \circ s$ with a strict epimorphism s and a strict monomorphism i.

Definition 2.1. — A *quasi-abelian* category is an additive category which admits kernels and cokernels and satisfies the following conditions:

(i) strict epimorphisms are stable by base changes,
(ii) strict monomorphisms are stable by co-base changes.

The condition (i) means that, for any strict epimorphism $u\colon X \to Y$ and any morphism $Y' \to Y$, setting $X' = X \times_Y Y' = \mathrm{Ker}(X \times Y' \to Y)$, the composition $X' \to X \times Y' \to Y'$ is a strict epimorphism. The condition (ii) is similar by reversing the arrows.

Note that, for any morphism $f\colon X \to Y$ in a quasi-abelian category, $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is both a monomorphism and an epimorphism.

Remark that if $\mathscr{C}$ is a quasi-abelian category, then its opposite category $\mathscr{C}^{\mathrm{op}}$ is also quasi-abelian.

Of course, an abelian category is a quasi-abelian category in which all morphisms are strict.

Definition 2.2. — Let $\mathscr{C}$ be a quasi-abelian category. A sequence $M' \xrightarrow{f} M \xrightarrow{f'} M''$ with $f' \circ f = 0$ is strictly exact if f is strict and the canonical morphism $\mathrm{Im}\, f \to \mathrm{Ker}\, f'$ is an isomorphism.

Equivalently such a sequence is strictly exact if the canonical morphism $\mathrm{Coim}\, f \to \mathrm{Ker}\, f'$ is an isomorphism.

One shall be aware that the notion of strict exactness is not auto-dual.

Consider a functor $F\colon \mathscr{C} \to \mathscr{C}'$ of quasi-abelian categories. Recall that F is

- strictly exact if it sends any strict exact sequence $X' \to X \to X''$ to a strict exact sequence,
- strictly left exact if it sends any strict exact sequence $0 \to X' \to X \to X''$ to a strict exact sequence $0 \to F(X') \to F(X) \to F(X'')$,
- left exact if it sends any strict exact sequence $0 \to X' \to X \to X'' \to 0$ to a strict exact sequence $0 \to F(X') \to F(X) \to F(X'')$.

Derived categories. — Let $\mathscr{C}$ be an additive category. One denotes as usual by $\mathrm{C}(\mathscr{C})$ the additive category consisting of complexes in $\mathscr{C}$. For $X \in \mathrm{C}(\mathscr{C})$, one denotes by X^n ($n \in \mathbb{Z}$) its n's component and by $d_X^n\colon X^n \to X^{n+1}$ the differential. For $k \in \mathbb{Z}$, one denotes by $X \mapsto X[k]$ the shift functor (also called translation functor). We denote by $\mathrm{C}^+(\mathscr{C})$ (resp. $\mathrm{C}^-(\mathscr{C})$, $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$) the full subcategory of $\mathrm{C}(\mathscr{C})$ consisting of objects X such that $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$, $|n| \gg 0$). One also sets $\mathrm{C}^{\mathrm{ub}}(\mathscr{C}) := \mathrm{C}(\mathscr{C})$ (ub stands for unbounded).

We do not recall here neither the construction of the mapping cone $\mathrm{Mc}(f)$ of a morphism f in $\mathrm{C}(\mathscr{C})$ nor the construction of the triangulated categories $\mathrm{K}^*(\mathscr{C})$ ($* = \mathrm{ub}, +, -, \mathrm{b}$), called the homotopy categories of $\mathscr{C}$.

Recall that a null system $\mathscr{N}$ in a triangulated category $\mathscr{T}$ is a full triangulated saturated subcategory of $\mathscr{T}$, saturated meaning that an object X belongs to $\mathscr{N}$ whenever X is isomorphic to an object of $\mathscr{N}$. For a null system $\mathscr{N}$, the localization $\mathscr{T}/\mathscr{N}$ is a triangulated category. A distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{T}/\mathscr{N}$ is a triangle isomorphic to the image of a distinguished triangle in $\mathscr{T}$.

Let $\mathscr{C}$ be quasi-abelian category. One says that a complex X is

- *strict* if all the differentials d_X^n are strict,
- *strictly exact in degree* n if the sequence $X^{n-1} \to X^n \to X^{n+1}$ is strictly exact.
- *strictly exact* if it is strictly exact in all degrees.

If X is strictly exact, then X is a strict complex and $0 \to \mathrm{Ker}(d_X^n) \to X^n \to \mathrm{Ker}(d_X^{n+1}) \to 0$ is strictly exact for all n.

Note that if two complexes X and Y are isomorphic in $\mathrm{K}(\mathscr{C})$, and if X is strictly exact, then so is Y. Let $\mathscr{E}$ be the full additive subcategory of $\mathrm{K}(\mathscr{C})$ consisting of strictly exact complexes. Then $\mathscr{E}$ is a null system in $\mathrm{K}(\mathscr{C})$.

Definition 2.3. — The derived category $\mathrm{D}(\mathscr{C})$ is the quotient category $\mathrm{K}(\mathscr{C})/\mathscr{E}$. where $\mathscr{E}$ is the null system in $\mathrm{K}(\mathscr{C})$ consisting of strictly exact complexes. One defines similarly the categories $\mathrm{D}^*(\mathscr{C})$ for $* = +, -, \mathrm{b}$.

A morphism $f\colon X \to Y$ in $\mathrm{K}(\mathscr{C})$ is called a *quasi-isomorphism* (a qis for short) if, after being embedded in a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, Z belongs to $\mathscr{E}$. This is equivalent to saying that its image in $\mathrm{D}(\mathscr{C})$ is an isomorhism. It follows that given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{K}(\mathscr{C})$, if two of f, g and $g \circ f$ are qis, then all the three are qis.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of morphisms in $\mathrm{C}(\mathscr{C})$ such that $0 \to X^n \to Y^n \to Z^n \to 0$ is strictly exact for all n, then the natural morphism $\mathrm{Mc}(f) \to Z$ is a qis, and we have a distinguished triangle

$$X \to Y \to Z \to X[1]$$

in $\mathrm{D}(\mathscr{C})$.

Left t-structure. — Let $\mathscr{C}$ be a quasi-abelian category. Recall that for $n \in \mathbb{Z}$, $\mathrm{D}^{\leq n}(\mathscr{C})$ (resp. $\mathrm{D}^{\geq n}(\mathscr{C})$) denotes the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of complexes X which are strictly exact in degrees $k > n$ (resp. $k < n$). Note that $\mathrm{D}^+(\mathscr{C})$ (resp. $\mathrm{D}^-(\mathscr{C})$) is the union of all the $\mathrm{D}^{\geq n}(\mathscr{C})$'s (resp. all the $\mathrm{D}^{\leq n}(\mathscr{C})$'s), and $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$ is the intersection $\mathrm{D}^+(\mathscr{C}) \cap \mathrm{D}^-(\mathscr{C})$. The associated truncation functors are then given by:

$$\begin{array}{lcccccccccc}
\tau^{\leq n}X\colon \cdots & \to & X^{n-2} & \to & X^{n-1} & \to & \operatorname{Ker} d_X^n & \to & 0 & \to & \cdots \\
\tau^{\geq n}X\colon \cdots & \to & 0 & \to & \operatorname{Coim} d_X^{n-1} & \to & X^n & \to & X^{n+1} & \to & \cdots .
\end{array}$$

The functor $\tau^{\leq n}\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^{\leq n}(\mathscr{C})$ is a right adjoint to the inclusion functor $\mathrm{D}^{\leq n}(\mathscr{C}) \hookrightarrow \mathrm{D}(\mathscr{C})$, and $\tau^{\geq n}\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^{\geq n}(\mathscr{C})$ is a left adjoint functor to the inclusion functor $\mathrm{D}^{\geq n}(\mathscr{C}) \hookrightarrow \mathrm{D}(\mathscr{C})$.

The pair $(\mathrm{D}^{\leq 0}(\mathscr{C}), \mathrm{D}^{\geq 0}(\mathscr{C}))$ defines a t-structure on $\mathrm{D}(\mathscr{C})$ by [**8**]. We refer to [**1**] for the general theory of t-structures (see also [**4**] for an exposition).

The heart $\mathrm{D}^{\leq 0}(\mathscr{C}) \cap \mathrm{D}^{\geq 0}(\mathscr{C})$ is an abelian category called the left heart of $\mathrm{D}(\mathscr{C})$ and denoted by $\mathrm{LH}(\mathscr{C})$ in [**8**]. The embedding $\mathscr{C} \hookrightarrow \mathrm{LH}(\mathscr{C})$ induces an equivalence

$$\mathrm{D}(\mathscr{C}) \xrightarrow{\sim} \mathrm{D}(\mathrm{LH}(\mathscr{C})).$$

By duality, one also defines the right t-structure and the right heart of $\mathrm{D}(\mathscr{C})$.

Derived functors. — Given an additive functor $F\colon \mathscr{C} \to \mathscr{C}'$ of quasi-abelian categories, its right or left derived functor is defined in [**8**, Def. 1.3.1] by the same procedure as for abelian categories.

Definition 2.4. — (See [**8**, Def. 1.3.2].) A full additive subcategory $\mathscr{P}$ of $\mathscr{C}$ is called F-projective if