383

ASTÉRISQUE

2016

SUBANALYTIC SHEAVES AND SOBOLEV SPACES

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Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque
383, 2016, p. 61–94

SOBOLEV SPACES AND SOBOLEV SHEAVES

by

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Abstract. — Sobolev spaces $H^s_{\mathrm{loc}}(M)$ on a real manifold M are classical objects of Analysis. In this paper, we assume that M is real analytic and denote by M_{sa} the associated subanalytic site, for which the open sets are the open relatively compact subanalytic subsets and the coverings are, roughly speaking, the finite coverings. For $s \in \mathbb{R}, s \leq 0$, we construct an object $\mathscr{H}^s$ of the derived category $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ of sheaves on M_{sa} with the property that if U is open in M_{sa} and has a Lipschitz boundary, then the object $\mathscr{H}^s(U) := \mathrm{R}\Gamma(U; \mathscr{H}^s)$ is concentrated in degree 0 and coincides with the classical Sobolev space $H^s(U)$. This construction is based on the results of S. Guillermou and P. Schapira in this volume.

Moreover, in the special case where the manifold M is of dimension 2, we will compute explicitly the complex $\mathscr{H}^s(U)$ and prove that it is always concentrated in degree 0, but is not necessarily a subspace of the space of distributions on U.

***Résumé* (Espaces de Sobolev et faisceaux de Sobolev).** — Soit M une variété analytique réelle. Le site sous-analytique M_{sa} est constitué des ouverts sous-analytiques relativement compacts de M, les recouvrements étant finis à extraction près. Pour $s \in \mathbb{R}$, soit $H^s_{\mathrm{loc}}(M)$ l'espace de Sobolev usuel sur M. Pour tout $s \in \mathbb{R}, s \leq 0$ nous construisons un objet $\mathscr{H}^s$ de la catégorie dérivée $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ des faisceau sur M_{sa}, qui vérifie la propriété suivante: pour tout ouvert $U \in M_{\mathrm{sa}}$ à frontière lipschitzienne, $\mathscr{H}^s(U) := \mathrm{R}\Gamma(U; \mathscr{H}^s)$ est concentré en degré 0 et coïncide avec l'espace de Sobolev usuel $H^s(U)$. Cette construction utilise les résultats de S. Guillermou et P. Schapira contenus dans ce volume.

Dans le cas où M est de dimension 2, nous explicitons le complexe $\mathscr{H}^s(U)$. Nous démontrons qu'il est toujours concentré en degré 0, mais ne s'identifie pas toujours à un sous-espace de distributions sur U.

2010 Mathematics Subject Classification. — 46E35, 16E35, 18F20, 58A03.
Key words and phrases. — Sobolev spaces, subanalytic topology, sheaves, derived categories.

This work was supported by the European Research Council, ERC-2012-ADG, project number 320845: Semi Classical Analysis of Partial Differential Equations.

1. Introduction

Let M be a real analytic manifold. Let us recall that for $s \in \mathbb{R}$, and $x_0 \in M$ one says that a distribution $u \in \mathscr{D}'(M)$ belongs to the space $H^s_{x_0}(M)$ iff there exists a properly supported pseudodifferential operator P of degree s, elliptic at x_0, such that $Pu \in L^2_{\mathrm{loc}}(M)$. As usual, we denote by $H^s_{\mathrm{loc}}(M)$ the space of distributions u on M such that $u \in H^s_{x_0}(M)$ for all $x_0 \in M$. For U open and relatively compact in M, we define the space $H^s(U)$ by

$$H^s(U) = \{f \in \mathscr{D}'(U),\ \exists g \in H^s_{\mathrm{loc}}(M),\ g|_U = f\}.$$

Following [3], we endow the real analytic manifold M with the subanalytic topology and denote by M_{sa} the site so-obtained. Recall that the open sets of this Grothendieck topology are the relatively compact open subanalytic subsets of M and the coverings are the finite coverings. As usual, one denotes by $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ the derived category of sheaves of $\mathbb{C}$-vector spaces on M_{sa} consisting of spaces bounded from below.

In this paper, we address the following question:

Let $s \in \mathbb{R}$ be given. Does there exists an object $\mathscr{H}^s$ of $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$, such that the following requirement holds true:

(1.1) If U is open, Lipschitz, and relatively compact, then the complex $\mathscr{H}^s(U)$ is concentrate in degree 0 and is equal to $H^s(U)$.

If 1.1 holds true, then we will say that the object $\mathscr{H}^s$ of $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ is a "Sobolev sheaf". Clearly, this problem depends on the parameter $s \in \mathbb{R}$. It turns out that the answer to the above question is a straightforward byproduct of a theorem of A. Parusinski [5] for the values $s \in]-1/2, 1/2[$. More precisely, for $s \in]-1/2, 1/2[$, $U \mapsto H^s(U)$ is a sheaf on M_{sa}, with cohomology concentrated in degree 0 (see Lemma 5.2 in Section 5).

In this paper, we will construct the Sobolev sheaf $\mathscr{H}^s$ for any $s \leq 0$; this construction is based on the results of S. Guillermou and P. Schapira in [1]. Moreover, in the special case where the manifold M is of dimension 2, we will compute explicitly the complex $\mathscr{H}^s(U)$ for any bounded subanalytic open subset of M; it turns out that in dimension 2 and for $s \leq 0$, $\mathscr{H}^s(U)$ is always concentrated in degree 0, but is not always a subspace of $\mathscr{D}'(U)$. We will address the existence of the Sobolev sheaf $\mathscr{H}^s$ for $s \geq 0$ in a forthcoming paper.

Let us recall that for any object $\mathscr{F}$ of $\mathsf{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$, if we denote by $\mathbb{H}^j(U, \mathscr{F})$ the jth cohomology space of the complex $\mathscr{F}(U)$, one has the exact long Mayer Vietoris sequence, where U, V are two open subanalytic relatively compact subsets of M

$$(1.2)\quad \to \mathbb{H}^j(U \cup V, \mathscr{F}) \to \mathbb{H}^j(U, \mathscr{F}) \oplus \mathbb{H}^j(V, \mathscr{F}) \to \mathbb{H}^j(U \cap V, \mathscr{F}) \to \mathbb{H}^{j+1}(U \cup V, \mathscr{F}) \to$$

The construction of a Sobolev sheaf $\mathscr{H}^s$ is a purely local problem near any point of M. In fact, all the Sobolev spaces introduced in this article are $C^\infty_0(M)$ modules. Hence we may and will assume $M = \mathbb{R}^n$ in all the paper.

The paper is organized as follows:

In Section 2, we recall some basic facts on Sobolev spaces on $\mathbb{R}^n$, and we introduce the spaces $H^s(U), H^s_0[U]$, $U \subset \mathbb{R}^n$ open, and the spaces H^s_F, $F \subset \mathbb{R}^n$ closed.

In Section 3, we study the Sobolev spaces $H^s(U)$ when U is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. The main result in this section is Proposition 3.6. From the requirement (1.1) and the exact long Mayer Vietoris sequence (1.2), the validity of Proposition 3.6 is a necessary condition for the existence of a Sobolev sheaf $\mathscr{H}^s$.

Section 4 is devoted to the study of the auxiliary spaces $X^t(U)$ and $Y^s(U)$. The main result in this section is Proposition 4.11 which implies that the sheaf $U \mapsto Y^s(U)$ is Γ-acyclic on the linear subanalytic site.

Section 5 is devoted to the construction of the Sobolev sheaf $\mathscr{H}^s$ in the case $s \leq 0$. In Subsection 5.1, using Proposition 4.11, the construction of the Sobolev sheaf $\mathscr{H}^s$ for $s \leq 0$ becomes a simple byproduct of the results of S. Guillermou and P. Schapira in [**1**]. In Subsection 5.2, we compute explicitly the cohomology of the complex $\mathscr{H}^s(U)$ on $\mathbb{R}^2$ for $s \leq 0$. In particular, we verify that this complex is in degree 0, but $\mathbb{H}^0(U, \mathscr{H}^s)$ is not always a subspace of $\mathscr{D}'(U)$.

Finally, in the appendix, we give in Section 6.1 some results about interpolation spaces, and we recall in Section 6.2 the "classical" definition of Sobolev spaces given in the book of Lions and Magenes [**4**], and their relations with our spaces.

In all the paper, we shall use the following notations:

$B(x,r) = \{y \in \mathbb{R}^n, |y-x| < r\}$ is the open Euclidean ball with center x and radius r.

For $s \in \mathbb{R}$, we denote by $[s]$ be the integer part of s and $\{s\} = s - [s] \in [0,1[$.

We will denote by $\mathbb{H}^{j,s}(U)$ the $j$ème cohomology space of the complex $\mathscr{H}^s(U)$.

Acknowledgement. — We warmly thank Pierre Schapira for the numerous discussions we have had about the possibility of sheafying the Sobolev spaces and defining them as complexes of sheaves, or more precisely as objects of a derived category.

2. Notations and basic results on Sobolev spaces

Let us first recall that for $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is the space of tempered distributions f such that the Fourier transform $\hat{f}$ is in L^2_{loc} and $(1+|\xi|^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R}^n)$. It is an Hilbert space with the norm

$$\|f\|^2_{H^s} = \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

Let us recall (see [**2**], Section 7.9) that for $s \geq 0$, with $k = [s]$ and $r = \{s\}$, one has $f \in H^s(\mathbb{R}^n))$ if and only if $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for all $\alpha, |\alpha| \leq k$, and (in the case $r > 0$), $\frac{\partial^\alpha f(x) - \partial^\alpha f(y)}{|x-y|^{n/2+r}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\alpha, |\alpha| = k$. Moreover, the square of the H^s norm is equivalent to

$$(2.1)\qquad \sum_{|\alpha|\leq k}\int_{\mathbb{R}^n}|\partial^\alpha f(x)|^2\,dx+\mathbf{1}_{r>0}\sum_{|\alpha|=k}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|^2}{|x-y|^{n+2r}}\,dxdy.$$

If F is a closed subset of $\mathbb{R}^n$, we denote by H^s_F the closed subspace of $H^s(\mathbb{R}^n)$

$$(2.2)\qquad H^s_F=\{f\in H^s(\mathbb{R}^n),\ \mathrm{support}(f)\subset F\}.$$

If U is an open subset of $\mathbb{R}^n$, we denote by $H^s_0[U]$ the closure of $C^\infty_0(U)$ for the topology of $H^s(\mathbb{R}^n)$. Obviously, $H^s_0[U]$ is a closed subspace of $H^s_{\overline{U}}$.

For U open in $\mathbb{R}^n$, we denote by $H^s(U)$ the subspace of $\mathscr{D}'(U)$

$$(2.3)\qquad H^s(U)=\{f\in\mathscr{D}'(U),\exists g\in H^s(\mathbb{R}^n),g|_U=f\}.$$

We put on $H^s(U)$ the quotient topology:

$$(2.4)\qquad \|f\|_{H^s(U)}=\inf(\|g\|_{H^s(\mathbb{R}^n)},\ g|_U=f).$$

Then one has the exact sequence

$$(2.5)\qquad 0\to H^s_{\mathbb{R}^n\setminus U}\to H^s(\mathbb{R}^n)\to H^s(U)\to 0$$

which defines an Hilbert structure on $H^s(U)$.

Remark 2.1. — *The definition of $H^s(U)$ given by* (2.3) *is not the "usual" definition of the Sobolev space on U given in* **[4]**. *However, we will see in Section 6.2 that when U is Lipschitz and bounded,* (2.3) *coincides with the usual definition for all values of $s\in\mathbb{R}$, except for $s=-1/2-k,k\in\mathbb{N}$. Observe also that with the definition* (2.3), *it is obvious that for any s and α, the derivation ∂^α maps $H^s(U)$ into $H^{s-|\alpha|}(U)$. However, (see Section 6.2, Lemma 6.6) the map $f\mapsto\partial_x f$ does not map (!) $H^{1/2}(]0,\infty[)$ into $H^{-1/2}(]0,\infty[)$ with the usual definition of $H^{-1/2}(]0,\infty[)$ given in* **[4]**.

Let U be an open subset of $\mathbb{R}^n$, $s\in\mathbb{R}$ and $t=-s$. There is a natural duality pairing between the spaces $H^s(U)$ and $H^t_0[U]$. It is defined for $f\in H^s(U)$ and $\psi\in H^t_0[U]$ by the formula

$$(2.6)\qquad <f,\psi>=\lim_{n\to\infty}<g,\psi_n>,\quad g\in H^s(\mathbb{R}^n),\ g|_U=f$$

where $\psi_n\in C^\infty_0(U)$ is a sequence which converges to ψ in $H^t(\mathbb{R}^n)$. One has obviously from the above definitions

$$(2.7)\qquad |<f,\psi>|\leq\|f\|_{H^s(U)}\|\psi\|_{H^t_0[U]}.$$

From (2.7), the canonical map j from $H^t_0[U]$ into the dual space of $H^s(U)$ defined by $j(\psi)(f)=<f,\psi>$ is continuous, and the map $\tilde{j}$ from $H^s(U)$ into the dual space of $H^t_0[U]$ defined by $\tilde{j}(f)(\psi)=<f,\psi>$ is continuous.

Lemma 2.2. — *The map j is an isomorphism of $H^t_0[U]$ onto the dual space of $H^s(U)$. The map $\tilde{j}$ is an isomorphism of $H^s(U)$ onto the dual space of $H^t_0[U]$.*