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SUBANALYTIC SHEAVES AND SOBOLEV SPACES

Regular subanalytic covers

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REGULAR SUBANALYTIC COVERS

by

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Abstract. — Let U be an open relatively compact subanalytic subset of a real analytic manifold M. We show that there exists a "finite linear covering" (in the sense of Guillermou-Schapira) of U by subanalytic open subsets of U homeomorphic to an open ball.

We also show that the characteristic function of U can be written as a finite linear combination of characteristic functions of open relatively compact subanalytic subsets of M homeomorphic, by subanalytic and bi-lipschitz maps, to an open ball.

***Résumé* (Ensembles réguliers sous-analytiques).** — Soit U un ouvert sous-analytique relativement compact d'une variété analytique réelle M. Nous montrons qu'il existe un « recouvrement linéaire fini » (au sens de Guillermou-Schapira) de U par des ouverts sous-analytiques homéomorphes à une boule ouverte.

Nous montrons aussi que la fonction caractéristique de U peut s'écrire comme une combinaison linéaire finie de fonctions caractéristiques d'ouverts sous-analytiques relativement compacts de M homéomorphes, par des applications sous-analytiques et bi-lipschitz, à une boule ouverte.

Let M be a real analytic manifold of dimension n. In this paper we study the subalgebra $\mathscr{S}(M)$ of integer valued functions on M generated by characteristic functions of relatively compact open subanalytic subsets of M (or equivalently by characteristic functions of compact subanalytic subsets of M). As we show this algebra is generated by characteristic functions of open subanalytic sets with Lipschitz regular boundaries. More precisely, we call a relatively compact open subanalytic subset $U \subset M$ *an open subanalytic Lipschitz ball* if its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of $\mathbb{R}^n$. Here we assume that M is equipped with a Riemannian metric. Any two such metrics are equivalent on relatively compact sets and hence the above definition is independent of the choice of a metric.

Theorem 0.1. — *The algebra $\mathscr{S}(M)$ is generated by characteristic functions of open subanalytic Lipschitz balls.*

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That is to say if U is a relatively compact open subanalytic subset of M then its characteristic function 1_U is a finite integral linear combination of characteristic functions $1_{W_1}, \dots, 1_{W_m}$, where the W_j are open subanalytic Lipschitz balls. Note that, in general, U cannot be covered by finitely many subanalytic Lipschitz balls, as it is easy to see for $\{(x,y) \in \mathbb{R}^2; y^2 < x^3, x < 1\}$, $M = \mathbb{R}^2$, due to the presence of cusps. Nevertheless we show the existence of a "regular" cover in the sense that we control the distance to the boundary.

Theorem 0.2. — *Let U be an open relatively compact subanalytic subset of M. Then there exist a finite cover $U = \bigcup_i U_i$ by open subanalytic sets such that :*

1. *every U_i is subanalytically homeomorphic to an open n-dimensional ball;*
2. *there is $C > 0$ such that for every $x \in U$, $\operatorname{dist}(x, M \setminus U) \leq C \max_i \operatorname{dist}(x, M \setminus U_i)$.*

The proof of Theorem 0.1 is based on the classical cylindrical decomposition and the L-regular decomposition of subanalytic sets, cf. [**4, 9**], [**10**]. L-regular sets are natural multidimensional generalization of classical cusps. We recall them briefly in Subsection 1.6. For the proof of Theorem 0.2 we need also the regular projection theorem, cf. [**7, 8**], [**9**], that we recall in Subsection 1.4.

We also show the following strengthening of Theorem 0.2.

Theorem 0.3. — *In Theorem 0.2 we may require additionally that all U_i are open L-regular cells.*

For an open $U \subset M$ we denote $\partial U = \overline{U} \setminus U$.

1. Proofs

1.1. Reduction to the case $M = \mathbb{R}^n$.— Let U be an open relatively compact subanalytic subset of M. Choose a finite cover $\overline{U} \subset \bigcup_i V_i$ by open relatively compact sets such that for each V_i there is an open neighborhood of $\overline{V}_i$ analytically diffeomorphic to $\mathbb{R}^n$. Then there are finitely many open subanalytic U_{ij} such that $U_{ij} \subset V_i$ and 1_U is a combination of $1_{U_{ij}}$. Thus it suffices to show Theorem 0.1 for relatively compact open subanalytic subsets of $\mathbb{R}^n$.

Similarly, it suffices to show Theorems 0.2 and 0.3 for $M = \mathbb{R}^n$. Indeed, it follows from the observation that the function

$$x \to \max_i \operatorname{dist}(x, M \setminus V_i)$$

is continuous and nowhere zero on $\bigcup_i V_i$ and hence bounded from below by a nonzero constant $c > 0$ on $\overline{U}$. Then

$$\operatorname{dist}(x, M \setminus U) \leq C_1 \leq c^{-1} C_1 \max_i \operatorname{dist}(x, M \setminus V_i)$$

where C_1 is the diameter of $\overline{U}$ and hence, if $c^{-1}C_1 \geq 1$,

$$\operatorname{dist}(x, M \setminus U) \leq c^{-1} C_1 \max_i (\min\{\operatorname{dist}(x, M \setminus U), \operatorname{dist}(x, M \setminus V_i)\}).$$

Now if for each $U \cap V_i$ we choose a cover $\bigcup_j U_{ij}$ satisfying the statement of Theorem 0.2 or 0.3 then for $x \in U$

$$\begin{aligned} \operatorname{dist}(x, M \setminus U) &\leq c^{-1} C_1 \max_i (\min\{\operatorname{dist}(x, M \setminus U), \operatorname{dist}(x, M \setminus V_i)\} \\ &\leq c^{-1} C_1 \max_i \operatorname{dist}(x, M \setminus U \cap V_i) \leq C c^{-1} C_1 \max_{ij} \operatorname{dist}(x, M \setminus U_{ij}). \end{aligned}$$

Thus the cover $\bigcup_{i,j} U_{ij}$ satisfies the claim of Theorem 0.2, resp. of Theorem 0.3.

1.2. Regular projections. — We recall after **[8, 9]** the subanalytic version of the regular projection theorem of T. Mostowski introduced originally in **[7]** for complex analytic sets germs.

Let $X \subset \mathbb{R}^n$ be subanalytic. For $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi_\xi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear projection parallel to $(\xi, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Fix constants $C, \varepsilon > 0$. We say that $\pi = \pi_\xi$ is *(C, ε)-regular at $x_0 \in \mathbb{R}^n$ (with respect to X)* if

(a) $\pi|_X$ is finite;

(b) the intersection of X with the open cone

$$\mathscr{C}_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbb{R} \setminus 0\} \tag{1.1}$$

is empty or a finite disjoint union of sets of the form

$$\{x_0 + \lambda_i(\eta)(\eta, 1); |\eta - \xi| < \varepsilon\},$$

where λ_i are real analytic nowhere vanishing functions defined on $|\eta - \xi| < \varepsilon$.

(c) the functions λ_i from (b) satisfy for all $|\eta - \xi| < \varepsilon$

$$\|\operatorname{grad} \lambda_i(\eta)\| \leq C |\lambda_i(\eta)|.$$

We say that $\mathscr{P} \subset \mathbb{R}^{n-1}$ defines *a set of regular projections for X* if there exists $C, \varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ there is $\xi \in \mathscr{P}$ such that π_ξ is (C, ε)-regular at x_0.

Theorem 1.1 **([8, 9])**. — *Let X be a compact subanalytic subset of $\mathbb{R}^n$ such that $\dim X < n$. Then the generic set of $n+1$ vectors $\xi_1, \ldots, \xi_{n+1}$, $\xi_i \in \mathbb{R}^{n-1}$, defines a set of regular projections for X.*

Here by generic we mean in the complement of a subanalytic nowhere dense subset of $(\mathbb{R}^{n-1})^{n+1}$.

1.3. Cylindrical decomposition. — We recall the first step of a basic construction called the cylindrical algebraic decomposition in semialgebraic geometry or the cell decomposition in o-minimal geometry, for details see for instance **[2, 3]**.

Set $X = \overline{U} \setminus U$. Then X is a compact subanalytic subset of $\mathbb{R}^n$ of dimension $n-1$. We denote by $Z \subset X$ the set of singular points of X that is the complement in X of the set

$$Reg(X) := \{x \in X; (X, x) \text{ is the germ of a real analytic submanifold of dimension } n-1\}.$$

Then Z is closed in X, subanalytic and $\dim Z \leq n-2$.

Assume that the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ restricted to X is finite. Denote by $\Delta_\pi \subset \mathbb{R}^{n-1}$ the union of $\pi(Z)$ and the set of critical values of $\pi|_{Reg(X)}$. Then Δ_π, called *the discriminant set of* π, is compact and subanalytic. It is clear that $\overline{\pi(U)} = \pi(U) \cup \Delta_\pi$.

Proposition 1.2. — *Let $U' \subset \pi(U) \setminus \Delta_\pi$ be open and connected. Then there are finitely many bounded real analytic functions $\varphi_1 < \varphi_2 < \cdots < \varphi_k$ defined on U', such that $X \cap \pi^{-1}(U')$ is the union of graphs of φ_i's. In particular, $U \cap \pi^{-1}(U')$ is the union of the sets*

$$\{(x', x_n) \in \mathbb{R}^n; x' \in U', \varphi_i(x') < x_n < \varphi_{i+1}(x')\},$$

and moreover, if U' is subanalytically homeomorphic to an open $(n-1)$-dimensional ball, then each of these sets is subanalytically homeomorphic to an open n-dimensional ball.

1.4. The case of a regular projection. — Fix $x_0 \in U$ and suppose that $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is (C, ε)-regular at $x_0 \in \mathbb{R}^n$ with respect to X. Then the cone (1.1) contains no point of Z. By [9] Lemma 5.2, this cone contains no critical point of $\pi|_{Reg(X)}$, provided ε is chosen sufficiently small (for fixed C). In particular, $x'_0 = \pi(x_0) \notin \Delta_\pi$.

In what follows we fix $C, \varepsilon > 0$ and suppose ε small. We denote the cone (1.1) by $\mathscr{C}$ for short. Then for $\tilde{C}$ sufficiently large, that depends only on C and ε, we have

$$\operatorname{dist}(x_0, X \setminus \mathscr{C}) \le \tilde{C} \operatorname{dist}(x'_0, \pi(X \setminus \mathscr{C})) \le \tilde{C} \operatorname{dist}(x'_0, \Delta_\pi). \tag{1.2}$$

The first inequality is obvious, the second follows from the fact that the singular part of X and the critical points of $\pi|_{Reg(X)}$ are both outside the cone.

1.5. Proof of Theorem 0.2. — Induction on n. Set $X = \overline{U} \setminus U$ and let $\pi_{\xi_1}, \dots, \pi_{\xi_{n+1}}$ be a set of (C, ε)-regular projections with respect to X. To each of these projections we apply the cylindrical decomposition. More precisely, let us fix one of these projections that for simplicity we suppose standard and denote it by π. Then we apply the inductive assumption to $\pi(U) \setminus \Delta_\pi$. Thus let $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$ be a finite cover satisfying the statement of Theorem 0.2. Applying to each U'_i Proposition 1.2 we obtain a family of cylinders that covers $U \setminus \pi^{-1}(\Delta_\pi)$. In particular they cover the set of those points of U at which π is (C, ε)-regular.

Lemma 1.3. — *Suppose π is (C, ε)-regular at $x_0 \in U$. Let U' be an open subanalytic subset of $\pi(U) \setminus \Delta_\pi$ such that $x'_0 = \pi(x_0) \in U'$ and*

$$\operatorname{dist}(x'_0, \Delta_\pi) \le \tilde{C} \operatorname{dist}(x'_0, \partial U'), \tag{1.3}$$

with $\tilde{C} \ge 1$ for which (1.2) holds. Then

$$\operatorname{dist}(x_0, X) \le (\tilde{C})^2 \operatorname{dist}(x_0, \partial U_1), \tag{1.4}$$

where U_1 is the member of cylindrical decomposition of $U \cap \pi^{-1}(U')$ containing x_0.