384 ASTÉRISQUE

2016

QUANTIZATIONS OF CONICAL SYMPLECTIC RESOLUTIONS

Quantizations of conical symplectic resolutions I: local and global structure

Tom Braden & Nicholas Proudfoot & Ben Webster

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque
384, 2016, p. 1–73

QUANTIZATIONS OF CONICAL SYMPLECTIC RESOLUTIONS I: LOCAL AND GLOBAL STRUCTURE

by

Tom Braden, Nicholas Proudfoot & Ben Webster

Abstract. — We re-examine some topics in representation theory of Lie algebras and Springer theory in a more general context, viewing the universal enveloping algebra as an example of the section ring of a quantization of a conical symplectic resolution. While some modification from this classical context is necessary, many familiar features survive. These include a version of the Beilinson-Bernstein localization theorem, a theory of Harish-Chandra bimodules and their relationship to convolution operators on cohomology, and a discrete group action on the derived category of representations, generalizing the braid group action on category $\mathcal{O}$ via twisting functors.

Our primary goal is to apply these results to other quantized symplectic resolutions, including quiver varieties and hypertoric varieties. This provides a new context for known results about Lie algebras, Cherednik algebras, finite W-algebras, and hypertoric enveloping algebras, while also pointing to the study of new algebras arising from more general resolutions.

Résumé **(Quantifications des résolutions symplectiques coniques I: structure locale et globale)**

Nous réexaminons certains sujets dans la théorie de la représentation de algèbres de Lie et théorie de Springer dans un contexte plus général, voyant l'algèbre enveloppante comme un exemple d'un anneau des sections d'un quantification d'une résolution symplectique conique. Alors que modification de ce contexte classique est nécessaire, beaucoup caractéristiques familiers survivons. Ceux-ci incluent une version de la théorème de localisation de Beilinson-Bernstein, une théorie de bimodules de Harish-Chandra et leur relation aux opérateurs de convolution sur cohomologie, et une action d'une groupe discrète sur la catégorie dérivée de représentations, en généralisant l'action de la groupe de tresses sur la catégorie $\mathcal{O}$ par foncteurs de twist.

Notre principal objectif est d'appliquer ces résultats à d'autres résolutions symplectiques quantifiées, y compris les variétés carquois et variétés hypertorique. Cela fournit un nouveau contexte pour les résultats connus sur algèbres de Lie, algèbres de Cherednik, algèbres W finies, et algèbres enveloppantes hypertoriques, tout en pointant à l'étude de nouvelles algèbres découlant des résolutions plus générales.

T.B. has been supported by NSA grants H98230-08-1-0097 and H98230-11-1-0180. N.P. has been supported by NSF grant DMS-0950383. B.W. has been supported by NSA grant H98230-10-1-0199.

1. Introduction

The dazzling success of algebraic geometry... has so much reorientated the field that one particular protagonist has suggested, no doubt with much justification, that enveloping algebras should now be relegated to a subdivision of the theory of rings of differential operators.

Anthony Joseph, *On the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra* [**33**]

In this paper, we argue against the relegation suggested above, in favor of a different geometric context. While viewing universal enveloping algebras as differential operators is unquestionably a powerful technique, the differential operators on flag varieties are odd men out in the world of differential operators as a whole. For example, the only known examples of projective varieties that are D-affine are homogeneous spaces for semi-simple complex Lie groups, and it is conjectured that no other examples exist. On the other hand, in this paper we consider a world where this special case is very much at home: quantizations of symplectic resolutions of affine singularities.

Differential operators on a smooth projective variety X form a deformation quantization of the cotangent bundle T^*X. If X is a homogeneous space for a semi-simple complex Lie group G, its cotangent bundle is a resolution of the closure of a nilpotent orbit in $\mathfrak{g}^*$ (or an affine variety finite over this one). If X is the flag variety, this is known as the *Springer resolution.* This is yet another sense in which these spaces are misfits; homogeneous spaces for semi-simple complex Lie groups are conjecturally the only examples of projective varieties whose cotangent bundles resolve affine singularities. For most projective varieties X, T^*X does not have enough global functions.

There are, however, many other examples of symplectic algebraic varieties that resolve affine cones. While the Springer resolution is the most famous, other examples include the minimal resolution of a Kleinian singularity, the Hilbert scheme of points on such a resolution, Nakajima quiver varieties, and hypertoric varieties. One can study deformation quantizations of these varieties, and many of them have the same affinity property enjoyed by the Springer resolution. This paper is a study of these deformation quantizations and their representation theory.

Several examples have been studied extensively by other authors. Universal enveloping algebras have been considered from an enormous number of angles for decades, and other examples such as spherical Cherednik algebras and finite W-algebras have been active fields of research for many years. The hypertoric case has recently been studied by Bellamy and Kuwabara [**11**] and by the authors of this paper, jointly with Licata [**16**]. On the other hand, very few works attempt to view all these examples in a single coherent theory. Kashiwara and Rouquier began to develop such a theory [**42**], and our paper might be regarded as a continuation of their work. A recent preprint of McGerty and Nevins [**46**] addresses similar issues, with results that are complementary to ours.

In Section 2, we discuss the algebraic geometry of conical symplectic resolutions; this is essentially all material already in the literature, but we collect it here for the convenience of the reader. Particularly important for us are deformations which appear in the work of Kaledin and Verbitsky; these show that any symplectic resolution flatly deforms to a smooth affine variety, which is key to many properties of its quantization. One ingredient we will use systematically is the conical structure: a choice of $\mathbb{C}^*$-action which makes the base into a cone and acts with positive weight on the symplectic form.

In Section 3, we discuss equivariant quantizations of a conical symplectic resolution $\mathfrak{M}$, which are classified by $H^2(\mathfrak{M};\mathbb{C})$ **[13, 45]**. We prove some basic results about the ring A of $\mathbb{S}$-invariant global sections, a filtered algebra whose associated graded is isomorphic to $\mathbb{C}[\mathfrak{M}]$. We also study the behavior of quantizations under (quantum) Hamiltonian reduction, proving a quantum version of the Duistermaat-Heckman theorem (Proposition 3.16).

In Section 4 we introduce the appropriate category $\mathscr{D}$-mod of modules over a quantization $\mathscr{D}$, which one may regard as the quantum analogue of the category of coherent sheaves (in particular, there is a finiteness assumption built into the definition). In the case where $\mathfrak{M}$ is a cotangent bundle, we show that this category is equivalent to the category of finitely generated twisted D-modules on the base, where the twist is determined by the period of the quantization. The rest of the section is dedicated to the study of the sections and localization functors that relate the category of modules over a quantization to the category of modules over the section ring A. We establish in Theorem 4.17 that these functors induce derived equivalences for generic periods.

Theorem A. — *Let $\mathfrak{M}$ be a conical symplectic resolution, and fix two classes $\eta, \lambda \in H^2(\mathfrak{M};\mathbb{C})$ such that η is the Chern class of an ample line bundle, or the strict transform of an ample line bundle on any other conical symplectic resolution of $\mathfrak{M}_0$. For all but finitely many complex numbers k, the quantization of $\mathfrak{M}$ with period $\lambda + k\eta$ is derived affine; that is, the derived functors of global sections and localization are inverse equivalences.*

In order to obtain an equivalence of abelian (rather than derived) categories that works for all (rather than only generic) periods, we replace the section ring A with a $\mathbb{Z}$-algebra, which mimics in a non-commutative setting the homogeneous coordinate ring of a projective variety. Given a quantized symplectic resolution along with a very ample line bundle, we construct a $\mathbb{Z}$-algebra Z and prove the following result (Theorem 5.8).

Theorem B. — *Let $\mathfrak{M}$ be a conical symplectic resolution, let $\mathscr{L}$ be a very ample line bundle on $\mathfrak{M}$, and let Z be the associated $\mathbb{Z}$-algebra. Then the category $\mathscr{D}$-mod is equivalent to the category of finitely generated modules over Z modulo the subcategory of bounded modules.*

Theorem B has three nice consequences. First, we use it to prove the following abelian analogue of Theorem A (Corollary 5.17).

Corollary B.1. — *Let $\mathfrak{M}$ be a conical symplectic resolution, and fix two classes $\eta, \lambda \in H^2(\mathfrak{M}; \mathbb{C})$ such that η is the Chern class of an ample line bundle. For all but finitely many positive integers k, the quantization of $\mathfrak{M}$ with period $\lambda + k\eta$ is affine; that is, the (abelian) functors of global sections and localization are inverse equivalences.*

Next, we prove a version of Serre's GAGA theorem [**63**]. More precisely, we consider the analytic quantization $\mathscr{D}^{\mathrm{an}}$ with the same period as $\mathscr{D}$, define the appropriate module category $\mathscr{D}^{\mathrm{an}}$-mod, and prove that it is equivalent to $\mathscr{D}$-mod (Theorem 5.22). The existing literature is fairly evenly divided between working in the algebraic and analytic categories, and this corollary is an indispensable tool that allows us to import previous results from both sides.

Corollary B.2. — *If $\mathfrak{M}$ is a conical symplectic resolution, then the analytification functor from $\mathscr{D}$-mod to $\mathscr{D}^{\mathrm{an}}$-mod is an equivalence of categories.*

Finally, we use Theorem B to prove a categorical version of Kirwan surjectivity, relating the category of equivariant modules on a quantization to the category of modules on the Hamiltonian reduction. We consider a restriction functor defined by Kashiwara and Rouquier, and we use our $\mathbb{Z}$-algebra formalism to construct left and right adjoints, thus proving that the restriction functor is essentially surjective (Theorem 5.31). In particular, this result establishes that our category $\mathscr{D}$-mod is the same as the analogous category considered by McGerty and Nevins (Remark 5.32). For a precise statement of the hypotheses of the following result, see the beginning of Section 5.5.

Corollary B.3. — *If $\mathfrak{M}$ is obtained via symplectic reduction from an action of a reductive group G on $\mathfrak{X}$, then every object of $\mathscr{D}$-mod extends to a twisted G-equivariant module over a quantization of $\mathfrak{X}$.*

Let $\mathfrak{M}_0 := \operatorname{Spec} \mathbb{C}[\mathfrak{M}]$ be the cone resolved by $\mathfrak{M}$, and consider the *Steinberg variety* $\mathfrak{Z} := \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$. The cohomology $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M})$ with supports in $\mathfrak{Z}$, which by Poincaré duality can be identified with the Borel-Moore homology group $H^{\mathrm{BM}}_{2\dim\mathfrak{M}}(\mathfrak{Z})$, has a natural algebra structure via convolution [**22**, §2.7]. Furthermore, if $\mathfrak{L} \subset \mathfrak{M}$ is a Lagrangian subscheme that is equal to the preimage of its image in $\mathfrak{L}_0 \subset \mathfrak{M}_0$, then the convolution algebra acts on $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$. In the special case where $\mathfrak{M} = T^*(G/B)$ and $\mathfrak{L}$ is the conormal variety to the Schubert stratification of G/B, the convolution algebra is isomorphic to the group algebra of the Weyl group, and $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$ is isomorphic to the regular representation. More generally, there is a natural algebra homomorphism from the group algebra $\mathbb{C}[W]$ of the Namikawa Weyl group of $\mathfrak{M}$ to the convolution algebra $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M})$.

Section 6 is devoted to categorifying the picture described in the paragraph above. The convolution algebra is replaced by the monoidal category of Harish-Chandra bimodules, which comes in both an algebraic and a geometric version. The module $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$ is replaced by a subcategory $\mathscr{C}^{\mathfrak{L}} \subset \mathscr{D}$-mod (respectively $C^{\mathfrak{L}_0} \subset A$-mod) which is a module category for the category of geometric (respectively algebraic)