384 ASTÉRISQUE

2016

QUANTIZATIONS OF CONICAL SYMPLECTIC RESOLUTIONS

Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality

Tom Braden & Anthony Licata & Nicholas Proudfoot & Ben Webster
(with an appendix by I. Losev)

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque
384, 2016, p. 75–179

QUANTIZATIONS OF CONICAL SYMPLECTIC RESOLUTIONS II: CATEGORY $\mathcal{O}$ AND SYMPLECTIC DUALITY

by

Tom Braden, Anthony Licata, Nicholas Proudfoot & Ben Webster
(with an appendix by I. Losev)

Abstract. — We define and study category $\mathcal{O}$ for a symplectic resolution, generalizing the classical BGG category $\mathcal{O}$, which is associated with the Springer resolution. This includes the development of intrinsic properties paralleling the BGG case, such as a highest weight structure and analogues of twisting and shuffling functors, along with an extensive discussion of individual examples.

We observe that category $\mathcal{O}$ is often Koszul, and its Koszul dual is often equivalent to category $\mathcal{O}$ for a different symplectic resolution. This leads us to define the notion of a symplectic duality between symplectic resolutions, which is a collection of isomorphisms between representation theoretic and geometric structures, including a Koszul duality between the two categories. This duality has various cohomological consequences, including (conjecturally) an identification of two geometric realizations, due to Nakajima and Ginzburg/Mirković-Vilonen, of weight spaces of simple representations of simply-laced simple algebraic groups.

An appendix by Ivan Losev establishes a key step in the proof that $\mathcal{O}$ is highest weight.

Résumé **(Quantifications des résolutions symplectiques coniques II: catégorie $\mathcal{O}$ et dualité symplectique)**

Nous définissons et etudions la catégorie $\mathcal{O}$ pour une résolution symplectique, généralisant la catégorie $\mathcal{O}$ classique de BGG, qui est associée à la résolution de Springer. Cela inclut le développement de propriétés intrinsèques en parallèle du cas de BGG, tels que la structure de plus haut poids et des analogues des foncteurs de twist et de battage, avec une discussion approfondie des exemples individuels.

Nous observons que la catégorie $\mathcal{O}$ est souvent Koszul, et son Koszul dual est souvent équivalent à la catégorie $\mathcal{O}$ pour une autre résolution symplectique. Cela nous amène à définir la notion de dualité symplectique entre les résolutions symplectiques, qui est une collection d'isomorphismes entre des structures de la théorie des représentations et géométrique, y compris une dualité de Koszul entre les deux catégories. Cette dualité a diverses conséquences cohomologiques, y compris (conjecturalement) une identification de deux réalisations géométriques, defini par Nakajima et Ginzburg/Mirković-Vilonen, des espaces de poids de simples représentations des groupes algébriques simples simplement lacées.

Une annexe par Ivan Losev établit une étape clé dans la preuve que $\mathcal{O}$ est de plus haut poids.

T.B. has been supported by NSA grants H98230-08-1-0097 and H98230-11-1-0180. A.L. has been supported by an ARC Discovery Early Career fellowship. N.P. has been supported by NSF grant DMS-0950383. B.W. has been supported by NSA grant H98230-10-1-0199.

1. Introduction

In this paper, we have two main goals:

- to introduce a version of category $\mathcal{O}$ attached to a symplectic variety with extra structure,
- to describe a conjectured relationship, which we call *symplectic duality*, between pairs of symplectic varieties. The most striking manifestation of this duality is a Koszul duality between the associated categories $\mathcal{O}$.

The motivating example is the classical BGG category $\mathcal{O}$, and the remarkable theorem of Beilinson, Ginzburg and Soergel [**8**] showing that a regular integral block of category $\mathcal{O}$ is Koszul self-dual. In our formulation, this means that the Springer resolution of the nilpotent cone is self-dual as a symplectic variety.

Our perspective throughout is to study the geometry of symplectic varieties using deformation quantizations and their representation theory. The specific varieties that we want to study are called *conical symplectic resolutions*. The prequel to this paper [**23**] introduced these varieties, their quantizations, and the categories of modules over these quantizations. Here we will concentrate on a particular subcategory of this module category: category $\mathcal{O}$.

Versions of category $\mathcal{O}$ have appeared in many places in the literature: for representations of $U(\mathfrak{g})$ in [**11**], for rational Cherednik algebras in [**51**], for W-algebras in [**28, 78**], and for hypertoric enveloping algebras in [**22**]. Our general definition includes all of these particular examples as particular cases, and we are able to prove many basic facts about these categories in a unified way. We will discuss the details of their structure further below.

There is a one striking observation about these categories that we wish to give special prominence: they are often standard Koszul, and Koszul dual to the category $\mathcal{O}$ attached to a different variety. This is the heart of our definition of symplectic duality; much of this paper is concerned with fleshing out the structures surrounding this observation and explaining how it looks in the various examples where it is known to hold.

We interpret symplectic duality as evidence of a hidden mirror symmetry-like connection between the two varieties, though at the moment it is difficult to make the nature of this connection mathematically precise. However, the same pairs of examples have arisen in moduli spaces of vacua for certain S-dual pairs of field theories in physics, suggesting this is not pure coincidence.

BGG category $\mathcal{O}$. Let us discuss the content of the paper in more detail. As mentioned above, our motivating example is the representation theory of $U(\mathfrak{g})$, whose geometric avatar is the Springer resolution of the nilcone by the cotangent bundle $T^*(G/B)$. Fix a regular class $\lambda \in \mathfrak{h}^* \cong H^2(G/B;\mathbb{C})$ and let $\mathcal{O}_{\mathrm{a}}$ be the subcategory of BGG category $\mathcal{O}$ consisting of modules over $U(\mathfrak{g})$ with the same generalized central character as the simple highest weight module with highest weight $\lambda - \rho$. The subscript stands for *algebraic*, since $\mathcal{O}_{\mathrm{a}}$ is defined as a category of modules over an algebra.

Let $\mathcal{O}_{\mathrm{g}}$ be the category of finitely generated $(\lambda - \rho)$-twisted D-modules [(1)] on G/B that are smooth with respect to the Schubert stratification. Here the subscript stands for *geometric*, since $\mathcal{O}_{\mathrm{g}}$ is defined as a category of sheaves. The following list gives some of the known structures and properties of the categories $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$. Our main goal in the paper will be to generalize these statements from $T^*(G/B)$ to arbitrary conical symplectic resolutions.

1. There exist localization and section functors relating $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$. These functors are always inverse derived equivalences, and they are Abelian equivalences if λ is dominant [**9**].
2. The two categories are both highest weight [**39**] and have graded lifts which are Koszul [**8**].
3. If λ is integral, then the center of the Yoneda algebra of $\mathcal{O}_{\mathrm{g}}$ is canonically isomorphic to $H^*(G/B;\mathbb{C})$ [**112**].
4. The Grothendieck group $K(\mathcal{O}_{\mathrm{g}})$ is isomorphic, via the characteristic cycle map, to the top Borel-Moore homology group of the union of the conormal varieties to the Schubert strata on G/B. This isomorphism intertwines the Euler form with a geometrically-defined intersection form.
5. The group $K(\mathcal{O}_{\mathrm{g}})$ decomposes as a direct sum over all nilpotent orbits by looking at microlocal supports of D-modules. The top Borel-Moore homology group of the union of the conormal varieties to the Schubert strata on G/B decomposes as a direct sum over all nilpotent orbits via the Beilinson-Bernstein-Deligne (BBD) decomposition [**6, 37**]. If λ is integral and $G \cong \mathrm{SL}_r$, then these two decompositions agree.
6. There are two collections of derived auto-equivalences of $\mathcal{O}_{\mathrm{a}}$, and of its graded lift, given by twisting and shuffling functors [**2, 61**]. These functors define two commuting actions of the Artin braid group of $\mathfrak{g}$ [**2, 83, 7**], and they categorify the left and right actions of the Weyl group on its group algebra.
7. The category $\mathcal{O}_{\mathrm{a}}$ is Koszul self-dual [**8**]. The induced derived auto-equivalence of the graded lift of $\mathcal{O}_{\mathrm{a}}$ exchanges twisting and shuffling functors [**83**, 6.5].

Category $\mathcal{O}$ in general. We now explain how these results generalize. Let $\mathfrak{M}_0$ be a Poisson cone, and let $\mathfrak{M} \to \mathfrak{M}_0$ be a symplectic resolution of $\mathfrak{M}_0$, equivariant with respect to the conical scaling action of $\mathbb{S} := \mathbb{C}^\times$. Let $\mathcal{D}$ be an $\mathbb{S}$-equivariant quantization of $\mathfrak{M}$, and let A be the ring of $\mathbb{S}$-invariant global sections of $\mathcal{D}$. Many rings of independent interest arise this way, such as spherical rational Cherednik algebras [**44**], central quotients of finite W-algebras [**100**], central quotients of hypertoric enveloping algebras [**10**], and (conjecturally) quotients of shifted Yangians [**69**] (see Section 2.3 for more details).

Let $\mathbb{T} := \mathbb{C}^\times$ act on $\mathfrak{M}$ by Hamiltonian symplectomorphisms that commute with $\mathbb{S}$, and assume that the fixed point set $\mathfrak{M}^{\mathbb{T}}$ is finite. The action of $\mathbb{T}$ on $\mathfrak{M}$ lifts to $\mathcal{D}$ and

(1) That is, modules over the sheaf of twisted differential operators denoted by $\mathscr{D}_\lambda$ in [**9**]; if λ is integral, this is simply the sheaf of differential operators on the line bundle with Euler class λ.

induces a $\mathbb{Z}$-grading on A. Let $A^+ \subset A$ be the non-negatively graded part. We define $\mathcal{O}_{\mathrm{a}}$ to be the category of finitely generated A-modules that are locally finite with respect to A^+. Versions of this category have already been studied for Cherednik algebras [**51, 106, 54**], for finite W-algebras [**28, 78, 118, 25**], and for hypertoric enveloping algebras [**21, 22**]. The classical case is where $\mathfrak{M} = T^*(G/B)$ and A is a central quotient of the universal enveloping algebra of $\mathfrak{g}$; if the period of the quantization is a regular element of $\mathfrak{h}^* \cong H^2(\mathfrak{M}; \mathbb{C})$, then $\mathcal{O}_{\mathrm{a}}$ is equivalent to the BGG category $\mathcal{O}_{\mathrm{a}}$ (Remark 3.11). (2)

Let

$$\mathfrak{M}^+ := \{p \in \mathfrak{M} \mid \lim_{\mathbb{T} \ni t \to 0} t \cdot p \text{ exists}\}.$$

We define $\mathcal{O}_{\mathrm{g}}$ to be the category of $\mathcal{D}$-modules that are set-theoretically supported on $\mathfrak{M}^+$ and admit a particularly nice lattice for a certain subalgebra $\mathcal{D}(0) \subset \mathcal{D}$; see Sections 2.5 and 3.15 for a precise definition. If $\mathfrak{M} = T^*(G/B)$ and $\mathbb{T}$ is a generic cocharacter of G, then $\mathfrak{M}^+$ is equal to the union of the conormal varieties to the Schubert strata, and $\mathcal{O}_{\mathrm{g}}$ is equivalent to the category $\mathcal{O}_{\mathrm{g}}$ above. The aforementioned results generalize as follows.

1. There exist localization and section functors relating $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$ (Corollary 3.19). These functors are inverse derived equivalences for most quantizations (Theorem 2.9), and they are Abelian equivalences if λ is sufficiently positive (Theorem 2.8).
2. The category $\mathcal{O}_{\mathrm{a}}$ is highest weight for most quantizations (Theorem 5.12 (3)), and $\mathcal{O}_{\mathrm{g}}$ is always highest weight (Proposition 5.17). We conjecture that both categories are Koszul (Conjectures 5.14 and 5.18). We can verify this conjecture in many examples, including cotangent bundles of partial flag varieties, S3-varieties, hypertoric varieties, Hilbert schemes on ALE spaces, and some quiver varieties (Section 9).
3. There is a natural graded ring homomorphism from $H^*(\mathfrak{M}; \mathbb{C})$ to the Yoneda algebra of $\mathcal{O}_{\mathrm{g}}$. We conjecture that, whenever $\mathcal{O}_{\mathrm{g}}$ is indecomposable (this will depend on the choice of quantization), this homomorphism will be an isomorphism (Conjecture 5.23). We can prove this conjecture for cotangent bundles of partial flag varieties, S3-varieties in type A, and hypertoric varieties (Section 9). We also formulate a stronger version of Conjecture 5.23, relating the equivariant cohomology of $\mathfrak{M}$ to the center of the universal deformation of the Yoneda algebra (Conjecture 10.32), which we prove in the latter two cases.
4. The Grothendieck group $K(\mathcal{O}_{\mathrm{g}})$ is isomorphic, via the characteristic cycle map, to the top Borel-Moore homology group of $\mathfrak{M}^+$. This isomorphism intertwines the Euler form on the Grothendieck group with the equivariant intersection form defined using the localization formula (Theorem 6.5).

(2) This statement really requires regularity of the period, otherwise it fails.
(3) The proof of this theorem relies heavily on an appendix by Ivan Losev.