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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS

A comparison of L-groups for covers of split reductive groups

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A COMPARISON OF L-GROUPS FOR COVERS OF SPLIT REDUCTIVE GROUPS

by

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Abstract. — In one article, the author has defined an L-group associated to a cover of a quasisplit reductive group over a local or global field. In another article, Wee Teck Gan and Fan Gao define (following an unpublished letter of the author) an L-group associated to a cover of a pinned split reductive group over a local or global field. In this short note, we give an isomorphism between these L-groups. In this way, the results and conjectures discussed by Gan and Gao are compatible with those of the author. Both support the same Langlands-type conjectures for covering groups.

Résumé **(Une comparaison des L-groupes pour les revêtements de groupes réductifs déployés)**
Dans un article, l'auteur a défini un L-groupe associé à un revêtement de groupes réductifs quasi-déployés sur un corps local ou global. Dans un autre article, Wee Teck Gan et Fan Gao définissent (suite à une lettre inédite de l'auteur) un L-groupe associé à un revêtement de groupes réductifs quasi-déployés sur un corps local ou global. Dans cette courte note, nous donnons un isomorphisme entre ces L-groupes. De cette manière, les résultats et les conjectures discutés par Gan et Gao sont compatibles avec ceux de l'auteur. Les deux soutiennent les mêmes conjectures de type Langlands pour les revêtements des groupes.

Summary of two constructions

Let $\mathbf{G}$ be a split reductive group over a local or global field F. Choose a Borel subgroup $\mathbf{B} = \mathbf{T}\mathbf{U}$ containing a split maximal torus $\mathbf{T}$ in $\mathbf{G}$. Let $X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ be the character lattice, and $Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ be the cocharacter lattice of $\mathbf{T}$. Let $\Phi \subset X$ be the set of roots and Δ the subset of simple roots. For each root $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the associated root subgroup. Let $\Phi^\vee$ and $\Delta^\vee$ be the associated coroots and simple coroots. The root datum of $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ is

$$\Psi = (X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee).$$

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Fix a pinning (épinglage) of $\mathbf{G}$ as well – a system of isomorphisms $x_\alpha \colon \mathbf{G}_a \to \mathbf{U}_\alpha$ for every root α.

The following notions of covering groups and their dual groups match those in [**3**]. Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ over F; in particular, $\#\mu_n(F) = n$. Here $\mathbf{G}'$ is a central extension of $\mathbf{G}$ by $\mathbf{K}_2$ in the sense of [**1**], and write $(Q, \mathscr{D}, f)$ for the three Brylinski-Deligne invariants of $\mathbf{G}'$. Assume that if n is odd, then $Q \colon Y \to \mathbb{Z}$ takes only even values (this is [**3**, Assumption 3.1]).

Let $\tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee$ be the dual group of $\tilde{\mathbf{G}}$, and let $\tilde{Z}^\vee$ be the center of $\tilde{G}^\vee$. The group $\tilde{G}^\vee$ is a pinned complex reductive group, associated to the root datum

$$(Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta}).$$

Here $Y_{Q,n} \subset Y$ is a sublattice containing nY. For each coroot $\alpha^\vee \in \Phi^\vee$, there is an associated positive integer n_α dividing n and a "modified coroot" $\tilde{\alpha}^\vee = n_\alpha \alpha^\vee \in \tilde{\Phi}^\vee$. The set $\tilde{\Phi}^\vee$ consists of the modified coroots, and $\tilde{\Delta}^\vee$ the modified simple coroots. Define $Y_{Q,n}^{\mathrm{sc}}$ to be the sublattice of $Y_{Q,n}$ generated by the modified coroots. Then

$$\tilde{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times) \text{ and } \tilde{Z}^\vee = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times).$$

Let $\bar{F}/F$ be a separable algebraic closure, and $\mathrm{Gal}_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. Fix an injective character $\epsilon \colon \mu_n(F) \hookrightarrow \mathbb{C}^\times$. From this data, the constructions of [**3**] and [**2**] both yield an L-group of $\tilde{\mathbf{G}}$ via a Baer sum of two extensions. In both papers, an extension

$$\tilde{Z}^\vee \hookrightarrow E_1 \twoheadrightarrow \mathrm{Gal}_F \qquad \text{(First twist)}$$

is described in essentially the same way. When F is local, this "first twist" E_1 is defined via a $\tilde{Z}^\vee$-valued 2-cocycle on Gal_F. See [**2**, §5.2] and [**3**, §5.4] (in the latter, E_1 is denoted $(\tau_Q)_* \widetilde{\mathrm{Gal}_F}$). Over global fields, the construction follows from the local construction and Hilbert reciprocity.

Both papers include a "second twist". Gan and Gao [**2**, §5.2] describe an extension

$$\tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}_F, \qquad \text{(Second twist)}$$

following an unpublished letter (June, 2012) from the author to Deligne. In [**3**], the second twist is the fundamental group of a gerbe, denoted $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{s})$. In this article $\bar{s} = \mathrm{Spec}(\bar{F})$, and so we write $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F})$ instead.

Both papers proceed by taking the Baer sum of these two extensions, $E = E_1 \dot{+} E_2$, to form an extension $\tilde{Z}^\vee \hookrightarrow E \twoheadrightarrow \mathrm{Gal}_F$. The extension E is denoted ${}^{\mathsf{L}}\tilde{Z}$ in [**3**, §5.4]. Then, one pushes out the extension E via $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$, to define the L-group

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_F\,. \qquad \text{(L-group)}$$

The two constructions of the L-group, from [**2**] and [**3**] are the same, except for insignificant linguistic differences, and a significant difference between the "second twists". In this short note, by giving an isomorphism,

$$\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F}) \text{ (described by the author)} \xrightarrow{\sim} E_2 \text{ (described by Gan and Gao)}$$

we will demonstrate that the second twists, and thus the L-groups, of both papers are isomorphic. Therefore, the work of Gan and Gao in [**2**] supports the broader conjectures of [**3**].

Remark 0.1. — Among the "insignificant linguistic differences," we note that Gan and Gao use extensions of $F^\times / F^{\times n}$ (for local fields) or the Weil group $\mathscr{W}_F$ rather than Gal_F. But pulling back via the reciprocity map of class field theory yields extensions of Gal_F by $\tilde{Z}^\vee$ as above.

1. Computations in the gerbe

1.1. Convenient base points. — Let $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$ be the gerbe constructed in [**3**, §3]. Rather than using the language of étale sheaves over F, we work with $\bar{F}$-points and trace through the Gal_F-action. Let $\hat{T} = \mathrm{Hom}(Y_{Q,n}, \bar{F}^\times)$ and $\hat{T}_{\mathrm{sc}} = \mathrm{Hom}(Y_{Q,n}^{\mathrm{sc}}, \bar{F}^\times)$. Let $p\colon \hat{T} \to \hat{T}_{\mathrm{sc}}$ be the surjective Gal_F-equivariant homomorphism dual to the inclusion $Y_{Q,n}^{\mathrm{sc}} \hookrightarrow Y_{Q,n}$. Define

$$\hat{Z} = \mathrm{Ker}(p) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \bar{F}^\times).$$

The reader is warned not to confuse $\hat{T}, \hat{T}_{\mathrm{sc}}, \hat{Z}$ with $\tilde{T}^\vee, \tilde{T}_{\mathrm{sc}}^\vee, \tilde{Z}^\vee$; the former are nontrivial Gal_F-modules (Homs into $\bar{F}^\times$) and the latter are trivial Gal_F-modules (Homs into $\mathbb{C}^\times$ as a trivial Gal_F-module).

Write $\bar{D} = \mathscr{D}(\bar{F})$ and $D = \mathscr{D}(F)$, where we recall $\mathscr{D}$ is the second Brylinski-Deligne invariant of the cover $\tilde{\mathbf{G}}$. We have a Gal_F-equivariant short exact sequence,

$$\bar{F}^\times \hookrightarrow \bar{D} \twoheadrightarrow Y.$$

By Hilbert's Theorem 90, the Gal_F-fixed points give a short exact sequence,

$$F^\times \hookrightarrow D \twoheadrightarrow Y.$$

Let $\bar{D}_{Q,n}$ and $\bar{D}_{Q,n}^{\mathrm{sc}}$ denote the preimages of $Y_{Q,n}$ and $Y_{Q,n}^{\mathrm{sc}}$ in $\bar{D}$. These are *abelian* groups, fitting into a commutative diagram with exact rows.

$$\begin{array}{ccccc} \bar{F}^\times & \hookrightarrow & \bar{D}_{Q,n}^{\mathrm{sc}} & \twoheadrightarrow & Y_{Q,n}^{\mathrm{sc}} \\ \downarrow{=} & & \downarrow & & \downarrow \\ \bar{F}^\times & \hookrightarrow & \bar{D}_{Q,n} & \twoheadrightarrow & Y_{Q,n}. \end{array}$$

Let $\mathrm{Spl}(\bar{D}_{Q,n})$ be the $\hat{T}$-torsor of splittings of $\bar{D}_{Q,n}$, and similarly let $\mathrm{Spl}(\bar{D}_{Q,n}^{\mathrm{sc}})$ be the $\hat{T}_{\mathrm{sc}}$-torsor of splittings of $\bar{D}_{Q,n}^{\mathrm{sc}}$.

Let $\overline{\mathrm{Whit}}$ denote the $\hat{T}_{\mathrm{sc}}$-torsor of nondegenerate characters of $\mathbf{U}(\bar{F})$. An element of $\overline{\mathrm{Whit}}$ is a homomorphism (defined over $\bar{F}$) from $\mathbf{U}$ to $\mathbf{G}_a$ which is nontrivial on every simple root subgroup $\mathbf{U}_\alpha$. Gal_F acts on $\overline{\mathrm{Whit}}$, and the fixed points $\mathrm{Whit} = \overline{\mathrm{Whit}}^{\mathrm{Gal}_F}$ are those homomorphisms from $\mathbf{U}$ to $\mathbf{G}_a$ which are defined over F. The $\hat{T}_{\mathrm{sc}}$-action on $\overline{\mathrm{Whit}}$ is described in [**3**, §3.3].

The pinning $\{x_\alpha : \alpha \in \Phi\}$ of $\mathbf{G}$ gives an element $\psi \in$ Whit. Namely, let ψ be the unique nondegenerate character of $\mathbf{U}$ which satisfies

$$\psi(x_\alpha(1)) = 1 \text{ for all } \alpha \in \Delta.$$

In [**3**, §3.3], we define an surjective homomorphism $\mu\colon \hat{T}_{\mathrm{sc}} \to \hat{T}_{\mathrm{sc}}$, and a Gal_F-equivariant isomorphism of $\hat{T}_{\mathrm{sc}}$-torsors,

$$\bar{\omega}\colon \mu_* \overline{\mathrm{Whit}} \to \mathrm{Spl}(D^{\mathrm{sc}}_{Q,n}).$$

The isomorphism $\bar{\omega}$ sends ψ to the unique splitting $s_\psi \in \mathrm{Spl}(D^{\mathrm{sc}}_{Q,n})$ which satisfies

$$s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha}, \text{ with } r_\alpha = (-1)^{Q(\alpha^\vee)\cdot \frac{n_\alpha(n_\alpha - 1)}{2}}.$$

We describe the element $[e_\alpha] \in D$ concisely here, based on [**1**, §11] and [**2**, §2.4]. Let $F(\!(v)\!)$ be the field of Laurent series with coefficients in F. The extension $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ splits over any unipotent subgroup, and so the pinning homomorphisms $x_\alpha\colon F(\!(v)\!) \to \mathbf{U}_\alpha(F(\!(v)\!))$ lift to homomorphisms

$$\tilde{x}_\alpha\colon F(\!(v)\!) \to \mathbf{U}'_\alpha(F(\!(v)\!)).$$

Define, for any $u \in F(\!(v)\!)^\times$,

$$\tilde{n}_\alpha(u) = \tilde{x}_\alpha(u)\tilde{x}_{-\alpha}(-u^{-1})\tilde{x}_\alpha(u).$$

This yields an element

$$\tilde{t}_\alpha = \tilde{n}_\alpha(v) \cdot \tilde{n}_\alpha(-1) \in \mathbf{T}'(F(\!(v)\!)).$$

Then t_α lies over $\alpha^\vee(v) \in \mathbf{T}(F(\!(v)\!))$. Its pushout via the tame symbol $\mathbf{K}_2(F(\!(v)\!)) \xrightarrow{\partial}\!\!\!\!\!\twoheadrightarrow F^\times$ is the element we call $[e_\alpha] \in D$.

Remark 1.1. — The element $s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha}$ coincides with what Gan and Gao call $s_{Q^{\mathrm{sc}}}(\tilde{\alpha}^\vee)$ in [**2**, §5.2]; the sign r_α arises from the formulae of [**1**, §11.1.4, 11.1.5].

Let $j_0\colon \hat{T}_{\mathrm{sc}} \to \mu_*\overline{\mathrm{Whit}}$ be the unique isomorphism of $\hat{T}_{\mathrm{sc}}$-torsors which sends 1 to ψ (or rather the image of ψ via $\overline{\mathrm{Whit}} \to \mu_*\overline{\mathrm{Whit}}$). Since $\psi \in$ Whit is Gal_F-invariant, this isomorphism j_0 is also Gal_F-invariant.

Finally, let $s \in \mathrm{Spl}(\bar{D}_{Q,n})$ be a splitting which restricts to s_ψ on $Y^{\mathrm{sc}}_{Q,n}$. Such a splitting s exists, since the map $\mathrm{Spl}(\bar{D}_{Q,n}) \to \mathrm{Spl}(\bar{D}^{\mathrm{sc}}_{Q,n})$ is surjective (since the map $\hat{T} \to \hat{T}_{\mathrm{sc}}$ is surjective). Note that s is not necessarily Gal_F-invariant (and often cannot be).

Let $h\colon \hat{T} \to \mathrm{Spl}(\bar{D}_{Q,n})$ be the function given by

$$h(x) = x^n * s \text{ for all } x \in \hat{T}.$$

The triple $\bar{z} = (\hat{T}, h, j_0)$ is an $\bar{F}$-object (i.e., a geometric base point) of the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$. Note that the construction of $\bar{z}$ depends on two choices: a pinning of $\mathbf{G}$ (to obtain $\psi \in$ Whit) and a splitting s of $\bar{D}_{Q,n}$ extending s_ψ. We call such a triple $\bar{z}$ a ***convenient base point*** for the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$.